# Multiagent Systems

1. [20 points] Find all Nash equilibria of the following game.

| | X | Y | Z |
|---|---|---|---|
| A | 2, −1 | 4, 2 | 2, 0 |
| B | 3, 3 | 0, 0 | 1, 1 |
| C | 1, 2 | 2, 8 | 5, 1 |

Sol)

- Pure strategy (10points)

Best response : $\begin{pmatrix} (2,-1) & (4,2) & (2,0) \\ (3,3) & (0,0) & (1,1) \\ (1,2) & (2,8) & (5,1) \end{pmatrix}$

**Answer : (A,Y), (B,X)**

- Mixed strategy (10points)

If we mix X and Y with equal prob. $\frac{1}{2}$, it strictly dominates Z. So, we can ignore Z.

Then, in reduced matrix, A strictly dominates C. So we ignore C.

Now, we consider reduce matrix

$$\begin{pmatrix} (2,-1) & (4,2) \\ (3,3) & (0,0) \end{pmatrix}$$

The following should be satisfied in mixed NE.

$$-p_1 + 3p_2 = 2p_1$$
$$p_1 + p_2 = 1$$

$$2q_1 + 4q_2 = 3q_1$$
$$q_1 + q_2 = 1$$

**Answer : (1/2, 1/2, 0), (4/5, 1/5, 0)**

2. [20 points] Consider any 2 × 2 matrix game in which neither player has a strategy dominated by a pure strategy. In such a game, is it possible for there to be a strategy dominated by a mixed strategy? Explain why or why not.

Sol)

**No**, it is not possible. Given that no strategy is dominated by a pure strategy, you need at least two strategies to be mixing in order to dominated a third. A mixed strategy in which A and B are played with positive probability dominates C if there exists a $\mathrm{p} \in (0,1)$ such that $\mathrm{p} * \mathrm{EU(A)} + (1 - \mathrm{p}) * \mathrm{EU(B)} > \mathrm{EU(C)}$ for any belief about the actions of the other player. Since we have only two strategies. C is either A or B. Simplifying the inequality, either $\mathrm{EU(B)} > \mathrm{EU(A)}$ or $\mathrm{EU(A)} > \mathrm{EU(B)}$ for any belief, which contradicts the fact that there is no pure-strategy dominance.

3. [20 points] Apply iterated elimination of dominated strategies to the following game. For each strategy that you eliminate, name the strategy that dominates it.

| | $X$ | $Y$ | $Z$ |
|---|---|---|---|
| $A$ | $3, 4$ | $3, 6$ | $4, 5$ |
| $B$ | $5, 1$ | $6, 1$ | $3, 2$ |
| $C$ | $1, 2$ | $1, 2$ | $5, 3$ |

Sol)

Iterated elimination steps:

- X is eliminated because it is dominated by Z
- A is eliminated because it is dominated by any mixed strategy (0,p,1-p), such that $p \in (\frac{2}{5}, \frac{1}{2})$ (you can use any member of this set as an example)
- Y is eliminated because it is dominated by Z
- B is eliminated because it is dominated by C
- Only C and Z remain
- **Answer : (C, Z) is the unique rationalizable strategy profile.**

4. [20 points] Explain how to exactly solve an assignment problem in polynomial time as detailed as possible. Explain also how to approximately solve an assignment problem by an auction-like algorithm as detailed as possible including the pseudocode of the algorithm.

Sol)

Assignment problem can be encoded as LP

$$
\begin{aligned}
\text{maximize} \quad & \sum_{(i,j)\in M} v(i,j)x_{i,j} & \\
\text{subject to} \quad & \sum_{j|(i,j)\in M} x_{i,j} \le 1 & \forall i \in N \\
& \sum_{i|(i,j)\in M} x_{i,j} \le 1 & \forall j \in X
\end{aligned}
$$

Furthermore, the optimal solution of the above encoded LP problem can be converted to integral solution with polynomial time, which yields the optimal solution of the original assignment problem. However, the polynomial-time solution to the LP problem is of complexity roughly $O(n^3)$ which may be too high in some cases. (10points)

Auction-like algorithm

**Naive Auction Algorithm**

// Initialization:
$S \leftarrow \emptyset$
**forall** $j \in X$ **do**
  $p_j \leftarrow 0$
**repeat**
  // Bidding Step:
  let $i \in N$ be an unassigned agent
  // Find an object $j \in X$ that offers $i$ maximal value at current prices:
  $j \in \arg\max_{k|(i,k)\in M}(v(i,k) - p_k)$
  // Compute $i$'s bid increment for $j$:
  $b_i \leftarrow (v(i,j) - p_j) - \max_{k|(i,k)\in M; k\neq j}(v(i,k) - p_k)$
  // which is the difference between the value to $i$ of the best and second-best objects at current prices (note that $i$'s bid will be the current price plus this bid increment).
  // Assignment Step:
  add the pair $(i,j)$ to the assignment $S$
  **if** *there is another pair* $(i',j)$ **then**
    remove it from the assignment $S$
  increase the price $p_j$ by the increment $b_i$
**until** $S$ *is feasible*  // that is, it contains an assignment for all $i \in N$

5. [20 points] Prove that if a solution F to a scheduling problem C is in competitive equilibrium at prices p, then F is also optimal for C.

Sol)

- **Definition 2.3.11 (Competitive equilibrium, generalized form)** Given a scheduling problem, a solution $F$ is in competitive equilibrium at prices p if and only if
  - For all $i \in N$ it is the case that $F_i = \underset{T \subseteq X}{\operatorname{argmax}}(v_i(T) - \sum_{j|x_j \in T} p_j)$ (the set of time slots allocated to agent $i$ maximizes his surplus at prices $p$);
  - For all $j$ such that $x_j \in F_0$, it is the case that $p_j = q_j$ (the price of all unallocated time slots is the reserve price); and
  - For all $j$ such that $x_j \notin F_0$ it is the case that $p_j \geq q_j$ (the price of all allocated time slots is greater than or equal to the reserve price)
- **Theorem 2.3.12** If a solution $F$ to a scheduling problem $C$ is in competitive equilibrium at prices $p$, then $F$ is also optimal for $C$
  - Show that the value of $F$ is higher than the value of any other solution $F'$
- **Theorem 2.3.13** A scheduling problem has a competitive equilibrium solution if and only if the LP relaxation of the associated integer program has a integer solution

$$
\begin{aligned}
V(F) &= \sum_{j|x_j \in F_0} q_j + \sum_{i \in N} v_i(F_i) \\
&= \sum_{j|x_j \in F_0} p_j + \sum_{i \in N} v_i(F_i) \\
&= \sum_{j|x_j \in X} p_j + \sum_{i \in N} \left[ v_i(F_i) - \sum_{j|x_j \in F_i} p_j \right] \\
&\geq \sum_{j|x_j \in X} p_j + \sum_{i \in N} \left[ v_i(F'_i) - \sum_{j|x_j \in F'_i} p_j \right] \\
&\geq \sum_{j|x_j \in F'_0} p_j + \sum_{i \in N} v_i(F'_i) \\
&\geq \sum_{j|x_j \in F'_0} q_j + \sum_{i \in N} v_i(F'_i) = V(F')
\end{aligned}
$$