Problem 1 (20 pts)

Solution)

Let us prove that X and Y are uncorrelated if they are independent first.

By definition 5.4,

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X,Y}(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)f_Y(y)dxdy$$
$$= \int_{-\infty}^{\infty} xf_X(x)dx \int_{-\infty}^{\infty} yf_Y(y)dy = E[X]E[Y]$$

Hence, $Cov[X,Y] = E[XY] - E[X]E[Y] = 0$, which implies X and Y are uncorrelated.

Note) *The independence of r.v.* X *and* Y *always implies their uncorrelatedness*

Then let us prove that X and Y are independent if they are uncorrelated.

If X and Y are uncorrelated $Cov[X,Y] = 0$ and their correlation coefficient $\rho_{X,Y}$ is 0.

From definition 5.10,

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y}\exp\left[-\frac{\left(\frac{X-\mu_X}{\sigma_X}\right)^2 + \left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2}{2}\right]$$

$$= \frac{1}{\sigma_X\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{X-\mu_X}{\sigma_X}\right)^2\right] \cdot \frac{1}{\sigma_Y\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2\right]$$

$$= f_X(x)f_Y(y),$$

which implies that X and Y are independent.

Note) *Uncorrelated* $\Rightarrow$ *Independent is not always true. Bivariate Gaussian distribution is a special case where it is true.*

Problem 2 (20 pts)

Solution)

Let X and Y are the amount of time that Romeo and Juliet is late at the date, respectively, Then what we have to compute is the PDF of r.v. $Z = |X - Y|$.

In order to compute $f_Z(z)$, we first compute $F_Z(z)$, the CDF of Z.

For $X - Y \geq 0$,

$$F_Z(z) = P(X - Y \leq z) = P(X \leq Y + z)$$

$$= \int_0^\infty \int_0^{y+z} f_{X,Y}(x,y)\,dxdy = \int_0^\infty \int_0^{y+z} \lambda\mu e^{-(\lambda x + \mu y)}\,dxdy$$

$$= \int_0^\infty \mu e^{-\mu y} \int_0^{y+z} \lambda e^{-\lambda x}\,dxdy = \int_0^\infty \mu e^{-\mu y}\left(1 - e^{-\lambda(y+z)}\right)dxdy$$

$$= 1 - \frac{\mu}{\lambda + \mu} e^{-\lambda z}$$

Hence we can get the PDF of Z by taking the derivative,

$$f_Z(z) = \frac{\lambda\mu}{\lambda + \mu} e^{-\lambda z}, \qquad z \geq 0 \qquad \cdots\cdots \text{(a)}$$

In the similar way, we can get the PDF of Z for $X - Y < 0$ as

$$f_Z(z) = \frac{\lambda\mu}{\lambda + \mu} e^{-\mu z}, \qquad z > 0 \qquad \cdots\cdots \text{(b)}$$

For $z > 0$, the complete PDF of r.v. Z is the summation of (a) and (b). In conclusion, the answer we are looking for is :

$$f_Z(z) = \begin{cases} \frac{\lambda\mu}{\lambda + \mu}(e^{-\lambda z} + e^{-\mu z}), & z > 0 \\ \frac{\lambda\mu}{\lambda + \mu}, & z = 0 \\ 0 & otherwise \end{cases}$$

Problem 3 (20 pts)

Solution)

By the definition of conditional probability,

$$P(X > s+t \quad | \ X > t) = \frac{P(X > s+t, X > t)}{P(X > t)} = \frac{P(X > s+t)}{P(X > t)} \qquad (\because s+t > t)$$
$$= \frac{1 - P(X < s+t)}{1 - P(X < t)} = \frac{1 - F_X(s+t)}{1 - F_X(t)}$$

The random variable X is an exponential random variable. Hence we can get the following.

$$F_X(s+t) = 1 - e^{-\lambda(s+t)}$$
$$F_X(t) = 1 - e^{-\lambda t}$$

$$\therefore \ P(X > s+t \quad | \ X > t)$$
$$= \frac{1 - F_X(s+t)}{1 - F_X(t)} = \frac{1 - \left(1 - e^{-\lambda(s+t)}\right)}{1 - (1 - e^{-\lambda t})} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}}$$
$$= e^{-\lambda s}$$
$$= 1 - \left(1 - e^{-\lambda s}\right)$$
$$= 1 - F_X(s) = P(X > s)$$

Problem 4 (20 pts)

Solution)

(a)

$$F_{U_n}(u) = \Pr(U_n \leq u) = \Pr\left(\cap_{i\in[1,n]} \{X_i \leq u\}\right) = (\Pr(X_i \leq u))^n = \left(F_x(u)\right)^n$$

(b)

$$F_{L_n}(l) = 1 - \Pr(L_n \geq l) = 1 - \Pr\left(\cap_{i\in[1,n]} \{X_i \geq l\}\right) = 1 - (1 - \Pr(X_i \leq l))^n$$

$$= 1 - \left(1 - F_x(l)\right)^n$$

(c)

$$F_{L_n,U_n}(l,u) = \Pr(L_n \leq l, U_n \leq u) = \Pr(U_n \leq u) - \Pr(L_n > l, U_n \leq u)$$

$$= \left(F_x(u)\right)^n - \Pr\left(\cap_{i\in[1,n]} l < X_i \leq u\right) = \left(F_x(u)\right)^n - (\max(0, \Pr(X_i \leq u) - \Pr(X_i \leq l)))^n$$

$$= \left(F_x(u)\right)^n - \left(\max\left(0, F_x(u) - F_x(l)\right)\right)^n$$

Problem 5 (20 pts)

Solution)

(a)

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \text{ and } f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}.$$

Because X and Y are independent,

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}}.$$

Using the fact that $X^2 + Y^2 = R^2$ and $dxdy = rdrd\theta$,

$$\begin{aligned} I &= \int_{x=0}^{\infty} \int_{y=0}^{\infty} f_{X,Y}(x,y) dydx = \int_{x=0}^{\infty} \int_{y=0}^{\infty} \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}} dydx \\ &= \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} \frac{1}{2\pi} r e^{-\frac{(r^2)}{2}} drd\theta = \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} f_{\Theta,R}(\theta,r) drd\theta. \end{aligned}$$

$$\therefore f_{\Theta,R}(\theta,r) = \frac{1}{2\pi} r e^{-\frac{(r^2)}{2}}.$$

By using marginal PDF,

$$\begin{aligned} f_R(r) &= \int_{\theta=0}^{2\pi} f_{\Theta,R}(\theta,r) d\theta = r e^{-\frac{(r^2)}{2}}, \\ f_\Theta(\theta) &= \int_{r=0}^{\infty} f_{\Theta,R}(\theta,r) dr = \left[-\frac{1}{2\pi} e^{-\frac{r^2}{2}} \right]_{r=0}^{\infty} = \frac{1}{2\pi}. \end{aligned}$$

Also, R and θ are independent because $f_{R,\Theta}(r,\theta) = f_R(r) f_\Theta(\theta)$.

(b)

From PDF of Problem (a)

$$F_R(r) = \int_{t=0}^{r} te^{-\frac{t^2}{2}} dt = \Big[- e^{-\frac{t^2}{2}} \Big]_{t=0}^{r} = 1 - e^{-\frac{r^2}{2}}, \forall r \geq 0.$$

Let's define $S = R^2$. Then, $dS = 2Rdr$ and CDF of S can be written as follows.

$$F_S(s) = P(S \leq s) = P(R \leq \sqrt{s}) = F_R(\sqrt{s}) = 1 - e^{-\frac{s}{2}}.$$

By differentiating (1), we can calculate the PDF of S.

$$f_S(s) = \frac{F_S(s)}{ds} = \frac{1}{2}e^{-\frac{1}{2}s},$$

which is the exponential distribution with parameter $\frac{1}{2}$.

Problem 6 (20 pts)

Solution)

$$X, Y, Z \sim U(0,1)$$

$$Let\ \ U = X + Y\ \ , for\ \ u \in [0,2]$$

$$f_U(u) = \int_{\max(0,u-1)}^{\min(1,u)} f_X(x) f_Y(u-x) dx$$

$$= \min(1,u) - \max(0,u-1)$$

$$= \begin{cases} 2-u & for & 2 \geq u > 1 \\ u & & 1 \geq u \geq 0 \end{cases}$$

$$Let\ \ W = U + Z, w \in [0,3],$$

$$f_W(w) = \int_{\max(0,w-1)}^{\min(2,w)} f_U(u) f_Z(w-u) du$$

for $3 \geq w > 2$

$$f_W(w) = \int_{w-1}^{2} 2 - u\ \ du = \left[2u - \frac{1}{2}u^2\right]_{w-1}^{2} = \frac{1}{2}(3-w)^2$$

for $2 \geq w > 1$

$$f_W(w) = \int_{w-1}^{w} f_U(u) du = \int_{1}^{w} 2 - u\ \ du + \int_{w-1}^{1} u\ \ du$$

$$= 3w - w^2 - \frac{3}{2}$$

for $1 \geq w \geq 0$

$$f_W(w) = \int_{0}^{w} u\ \ du = \frac{1}{2}w^2$$