Problem 1.

The solution to this problem is essentially the same as the proof of Theorem 13.13 except integrals are replaced by sums. First we verify that $\overline{X}_m$ is unbiased:

$$\mathsf{E}\left[\overline{X}_m\right] = \frac{1}{2m+1}\mathsf{E}\left[\sum_{n=-m}^{m} X_n\right] = \frac{1}{2m+1}\sum_{n=-m}^{m}\mathsf{E}\left[X_n\right] = \frac{1}{2m+1}\sum_{n=-m}^{m}\mu_X = \mu_X. \tag{1}$$

To show consistency, it is sufficient to show that $\lim_{m\to\infty}\mathsf{Var}[\overline{X}_m] = 0$. First, we observe that $\overline{X}_m - \mu_X = \frac{1}{2m+1}\sum_{n=-m}^{m}(X_n - \mu_X)$. This implies

$$\begin{aligned}
\mathsf{Var}[\overline{X}(T)] &= \mathsf{E}\left[\left(\frac{1}{2m+1}\sum_{n=-m}^{m}(X_n-\mu_X)\right)^2\right] \\
&= \mathsf{E}\left[\frac{1}{(2m+1)^2}\left(\sum_{n=-m}^{m}(X_n-\mu_X)\right)\left(\sum_{n'=-m}^{m}(X_{n'}-\mu_X)\right)\right] \\
&= \frac{1}{(2m+1)^2}\sum_{n=-m}^{m}\sum_{n'=-m}^{m}\mathsf{E}\left[(X_n-\mu_X)(X_{n'}-\mu_X)\right] \\
&= \frac{1}{(2m+1)^2}\sum_{n=-m}^{m}\sum_{n'=-m}^{m}C_X\left[n'-n\right].
\end{aligned} \tag{2}$$

We note that

$$\begin{aligned}
\sum_{n'=-m}^{m}C_X\left[n'-n\right] &\le \sum_{n'=-m}^{m}\left|C_X\left[n'-n\right]\right| \\
&\le \sum_{n'=-\infty}^{\infty}\left|C_X\left[n'-n\right]\right| = \sum_{k=-\infty}^{\infty}|C_X(k)| < \infty.
\end{aligned} \tag{3}$$

Hence there exists a constant K such that

$$\mathsf{Var}[\overline{X}_m] \le \frac{1}{(2m+1)^2}\sum_{n=-m}^{m}K = \frac{K}{2m+1}. \tag{4}$$

Thus $\lim_{m\to\infty}\mathsf{Var}[\overline{X}_m] \le \lim_{m\to\infty}\frac{K}{2m+1} = 0$.

Problem 2.

(a) Since $X(t)$ and $Y(t)$ are independent processes,

$$\mathsf{E}[W(t)] = \mathsf{E}[X(t)Y(t)] = \mathsf{E}[X(t)]\,\mathsf{E}[Y(t)] = \mu_X\mu_Y. \tag{1}$$

In addition,

$$\begin{aligned} R_W(t,\tau) &= \mathsf{E}[W(t)W(t+\tau)] \\ &= \mathsf{E}[X(t)Y(t)X(t+\tau)Y(t+\tau)] \\ &= \mathsf{E}[X(t)X(t+\tau)]\,\mathsf{E}[Y(t)Y(t+\tau)] \\ &= R_X(\tau)R_Y(\tau). \end{aligned} \tag{2}$$

We can conclude that $W(t)$ is wide sense stationary.

(b) To examine whether $X(t)$ and $W(t)$ are jointly wide sense stationary, we calculate

$$R_{WX}(t,\tau) = \mathsf{E}[W(t)X(t+\tau)] = \mathsf{E}[X(t)Y(t)X(t+\tau)]. \tag{3}$$

By independence of $X(t)$ and $Y(t)$,

$$R_{WX}(t,\tau) = \mathsf{E}[X(t)X(t+\tau)]\,\mathsf{E}[Y(t)] = \mu_Y R_X(\tau). \tag{4}$$

Since $W(t)$ and $X(t)$ are both wide sense stationary and since $R_{WX}(t,\tau)$ depends only on the time difference τ, we can conclude from Definition 13.18 that $W(t)$ and $X(t)$ are jointly wide sense stationary.

Problem 3.

(Sol) By definition, the stochastic process $\{X_t\}_{t\geq 0}$ is Gaussian if $(X_{t1}, X_{t2}, \dots, X_{tN})$ is a Gaussian vector for any choice of t_i. And $\mathrm{X} = (X_1, \dots, X_N)$ is a Gaussian vector if any linear combination $\sum_{i=1}^{N} k_i X_i$ is a Gaussian random variable. Hence it is enough to show that $\sum_{i=1}^{N} k_i W(t_i)$ is Gaussian for any choice of $\{k_i, t_i\}$, where $W(t)$ denotes Brownian process.

We have

$$\sum_{i=1}^{N} k_i W(t_i) = \sum_{i=1}^{N} k_i (W(t_i) - W(0))$$

since $W(0) = 0$. By the property of Brownian motion, $W(t_i) - W(0) \sim Gaussian$. Hence $\sum_{i=1}^{N} k_i W(t_i)$ is nothing but a linear combination of Gaussian random variable. $\therefore$ It is a Gaussian random vector and the Brownian process $W(t)$ is a Gaussian random process.

Problem 4.

Writing $Y(t+\tau) = \int_0^{t+\tau} N(v)\,dv$ permits us to write the autocorrelation of $Y(t)$ as

$$\begin{aligned} R_Y(t,\tau) = \mathrm{E}\left[Y(t)Y(t+\tau)\right] &= \mathrm{E}\left[\int_0^t \int_0^{t+\tau} N(u)N(v)\,dv\,du\right] \\ &= \int_0^t \int_0^{t+\tau} \mathrm{E}\left[N(u)N(v)\right]\,dv\,du \\ &= \int_0^t \int_0^{t+\tau} \alpha\delta(u-v)\,dv\,du. \end{aligned} \tag{1}$$

At this point, it matters whether $\tau \geq 0$ or if $\tau < 0$. When $\tau \geq 0$, then v ranges from 0 to $t+\tau$ and at some point in the integral over v we will have $v = u$. That is, when $\tau \geq 0$,

$$R_Y(t,\tau) = \int_0^t \alpha\,du = \alpha t. \tag{2}$$

When $\tau < 0$, then we must reverse the order of integration. In this case, when the inner integral is over u, we will have $u = v$ at some point so that

$$R_Y(t,\tau) = \int_0^{t+\tau} \int_0^t \alpha\delta(u-v)\,du\,dv = \int_0^{t+\tau} \alpha\,dv = \alpha(t+\tau). \tag{3}$$

Thus we see the autocorrelation of the output is

$$R_Y(t,\tau) = \alpha \min\{t, t+\tau\}. \tag{4}$$

Perhaps surprisingly, $R_Y(t,\tau)$ is what we found in Example 13.14 to be the autocorrelation of a Brownian motion process. In fact, Brownian motion is the integral of the white noise process.

Problem 5.

Let $W_n = X_1 + \cdots + X_n$. Since $M_n(X) = W_n/n$, we can write

$$\mathrm{P}\left[M_n(X) \geq c\right] = \mathrm{P}\left[W_n \geq nc\right]. \tag{1}$$

Since $\phi_{W_n}(s) = (\phi_X(s))^n$, applying the Chernoff bound to W_n yields

$$\mathrm{P}\left[W_n \geq nc\right] \leq \min_{s\geq 0} e^{-snc}\phi_{W_n}(s) = \min_{s\geq 0} \left(e^{-sc}\phi_X(s)\right)^n. \tag{2}$$

For $y \geq 0$, y^n is a nondecreasing function of y. This implies that the value of s that minimizes $e^{-sc}\phi_X(s)$ also minimizes $(e^{-sc}\phi_X(s))^n$. Hence

$$\mathrm{P}\left[M_n(X) \geq c\right] = \mathrm{P}\left[W_n \geq nc\right] \leq \left(\min_{s\geq 0} e^{-sc}\phi_X(s)\right)^n. \tag{3}$$

Problem 6.

(a) First $b = c$ since a covariance matrix is always symmetric. Second, $a = \text{Var}[X_1]$ and $b = \text{Var}[X_2]$. Hence we must have $a > 0$ and $d > 0$. Third, $\mathbf{C}$ must be positive definite, i.e. the eigenvalues of $\mathbf{C}$ must be positive. This can be tackled directly from first principles by solving for the eigenvalues using $\det(()\mathbf{C} - \lambda\mathbf{I}) = 0$. If you do this, you will find, after some algebra that the eigenvalues are

$$\lambda = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4b^2}}{2}. \tag{1}$$

The requirement $\lambda > 0$ holds iff $b^2 < ad$. As it happens, this is precisely the same conditon as requiring the correlation coefficient to have magnitude less than 1:

$$|\rho_{X_1X_2}| = \left|\frac{b}{\sqrt{ad}}\right| < 1. \tag{2}$$

To summarize, there are four requirements:

$$a > 0, \qquad d > 0, \qquad b = c, \qquad b^2 < ad. \tag{3}$$

(b) This is easy: for Gaussian random variables, zero covariance implies X_1 and X_2 are independent. Hence the answer is $b = 0$.

(c) X_1 and X_2 are identical if they have the same variance: $a = d$.

Problem 7.

(Sol)

$$\begin{aligned}
\underset{X}{\mathrm{E}}\left[Y \middle| N=n\right] &= \underset{X}{\mathrm{E}}\left[X_1+\ldots+X_N \middle| N=n\right] \\
&= \underset{X}{\mathrm{E}}\left[X_1+\ldots+X_N\right] \ (\because\ X_i \text{ and } N \text{ are independent, } i=1,\ldots,N) \\
&= \underset{X}{\mathrm{E}}[X_1]+\underset{X}{\mathrm{E}}[X_2]+\ldots+\underset{X}{\mathrm{E}}[X_N] \ (\because\ X_i \text{ and } X_j \text{ are independent, } i \neq j) \\
&= n\underset{X}{\mathrm{E}}[X] \ (\because\ \mathrm{E}[X_i]=\mathrm{E}[X],\ i=1,\ldots,N)
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}[Y] &= \underset{N}{\mathrm{E}}\left[N\underset{X}{\mathrm{E}}[X]\right] \\
&= \sum_{n=1}^{N} n\underset{X}{\mathrm{E}}[X]P_N(n) \\
&= \underset{X}{\mathrm{E}}[X]\sum_{n=1}^{N} nP_N(n) \\
&= \underset{X}{\mathrm{E}}[X]\underset{N}{\mathrm{E}}[N]
\end{aligned}$$

$$\therefore \mathrm{E}[Y] = \mathrm{E}[X]\mathrm{E}[N]$$