Section 13.1

Definitions and Examples

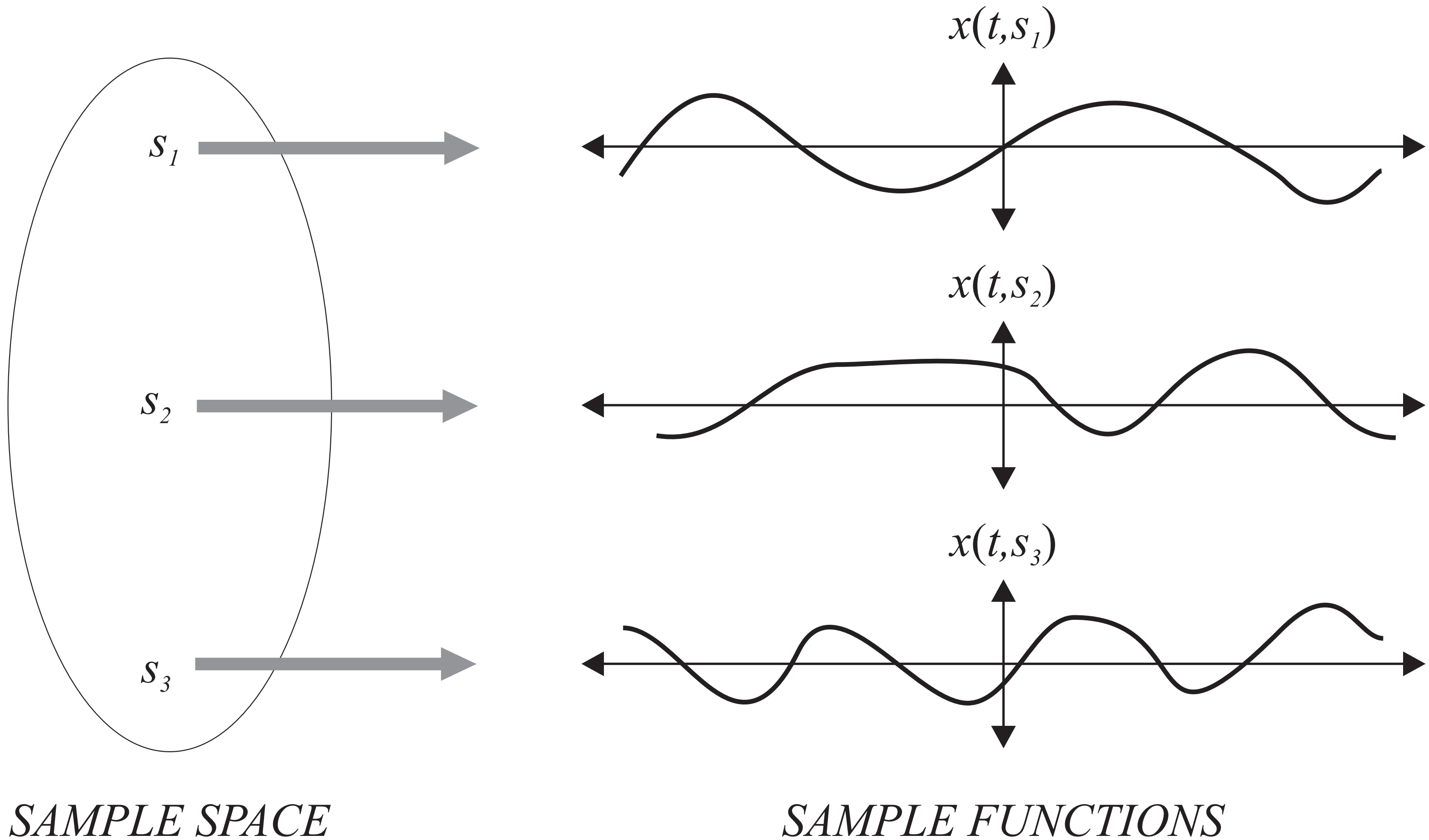
Definition 13.1 Stochastic Process

A stochastic process $X(t)$ *consists of an experiment with a probability measure* P$[\cdot]$ *defined on a sample space* S *and a function that assigns a time function* $x(t,s)$ *to each outcome* s *in the sample space of the experiment.*

Definition 13.2 Sample Function

A sample function $x(t,s)$ *is the time function associated with outcome* s *of an experiment.*

Figure 13.1



Conceptual representation of a random process.

Definition 13.3 Ensemble

The ensemble *of a stochastic process is the set of all possible time functions that can result from an experiment.*

Example 13.1

Starting at launch time $t = 0$, let $X(t)$ denote the temperature in Kelvins on the surface of a space shuttle. With each launch s, we record a temperature sequence $x(t, s)$. The ensemble of the experiment can be viewed as a catalog of the possible temperature sequences that we may record. For example,

$$x(8073.68, 175) = 207 \tag{13.1}$$

indicates that in the 175th entry in the catalog of possible temperature sequences, the temperature at $t = 8073.68$ seconds after the launch is 207 K.

Example 13.2

In Example 13.1 of the space shuttle, over all possible launches, the average temperature after 8073.68 seconds is $\mathrm{E}[X(8073.68)] = 217$ K. This is an ensemble average taken over all possible temperature sequences. In the 175th entry in the catalog of possible temperature sequences, the average temperature over that space shuttle mission is

$$\frac{1}{671,208.3}\int_0^{671,208.3} x(t,175)\,dt = 187.43 \text{ K}, \tag{13.2}$$

where the integral limit 671, 208.3 is the duration in seconds of the shuttle mission.

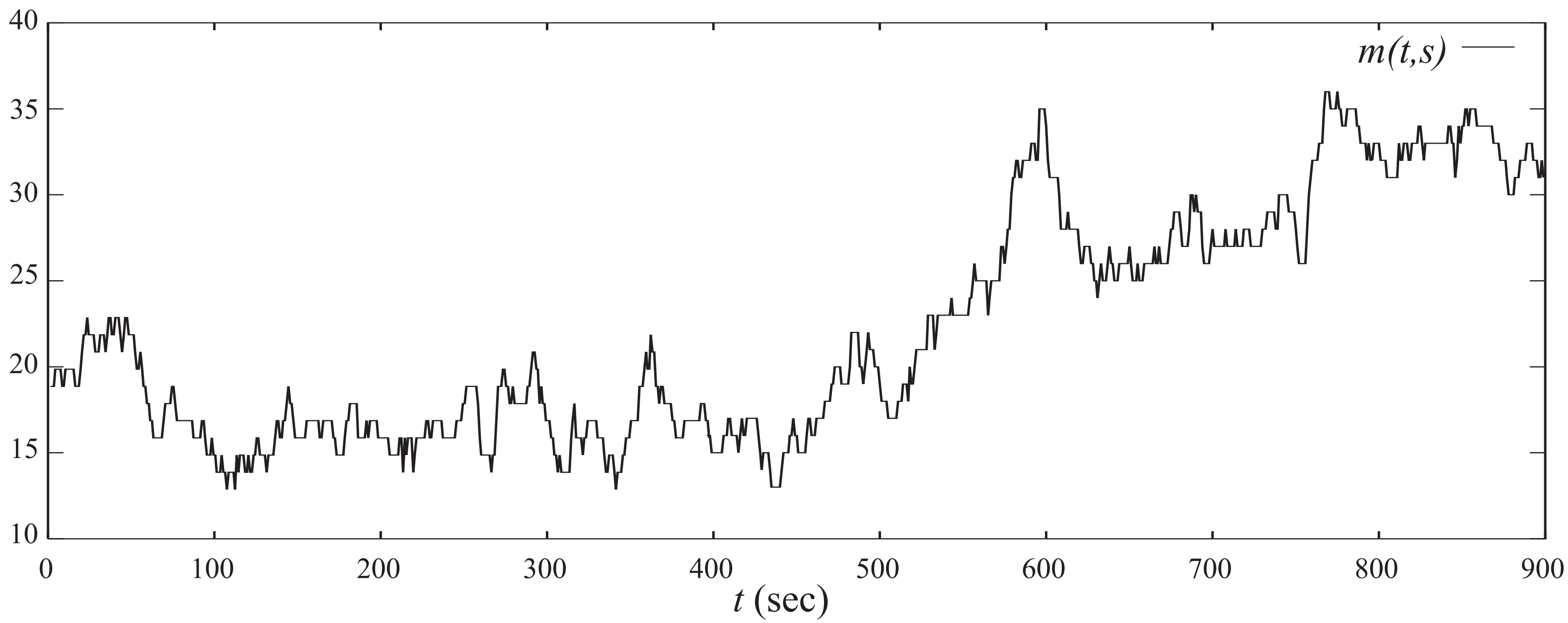
Example 13.3

Starting on January 1, we measure the noontime temperature (in degrees Celsius) at Newark Airport every day for one year. This experiment generates a sequence, $C(1), C(2), \ldots, C(365)$, of temperature measurements. With respect to the two kinds of averages of stochastic processes, people make frequent reference to both ensemble averages, such as "the average noontime temperature for February 19," and time averages, such as the "average noontime temperature for 1986."

Example 13.4

Consider an experiment in which we record $M(t)$, the number of active calls at a telephone switch at time t, at each second over an interval of 15 minutes. One trial of the experiment might yield the sample function $m(t, s)$ shown in Figure 13.2. Each time we perform the experiment, we would observe some other function $m(t, s)$. The exact $m(t, s)$ that we do observe will depend on many random variables including the number of calls at the start of the observation period, the arrival times of the new calls, and the duration of each call. An ensemble average is the average number of calls in progress at $t = 403$ seconds. A time average is the average number of calls in progress during a specific 15-minute interval.

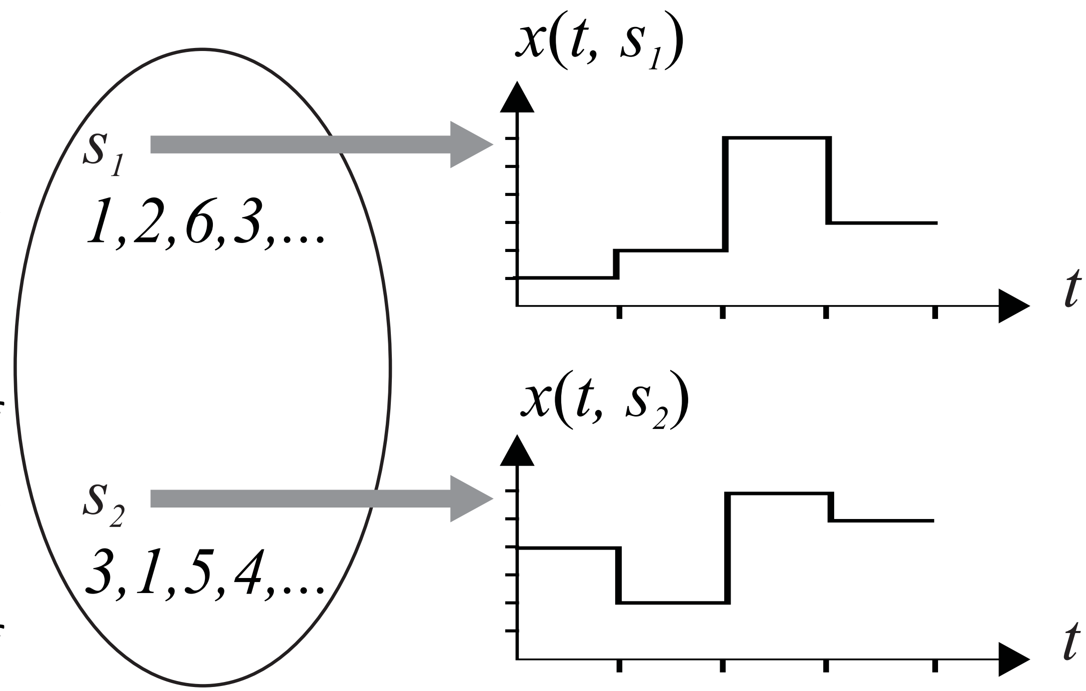
Figure 13.2



A sample function $m(t, s)$ of the random process $M(t)$ described in Example 13.4.

Example 13.5

Suppose that at time instants $T = 0, 1, 2, \ldots$, we roll a die and record the outcome N_T where $1 \leq N_T \leq 6$. We then define the random process $X(t)$ such that for $T \leq t < T + 1$, $X(t) = N_T$. In this case, the experiment consists of an infinite sequence of rolls and a sample function is just the waveform corresponding to the particular sequence of rolls. This mapping is depicted on the right.



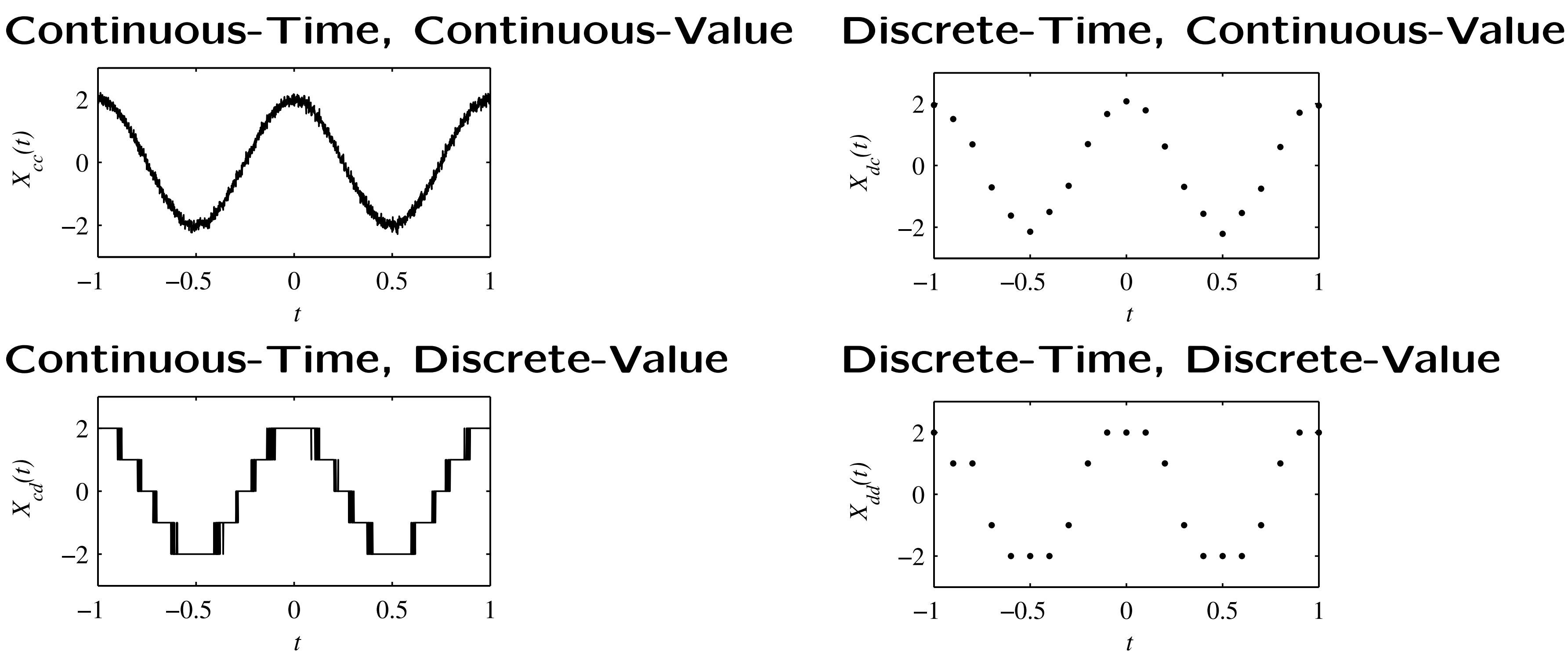
Example 13.6

The observations related to the waveform $m(t,s)$ in Example 13.4 could be

- $m(0,s)$, the number of ongoing calls at the start of the experiment,
- $X_1,\ldots,X_{m(0,s)}$, the remaining time in seconds of each of the $m(0,s)$ ongoing calls,
- N, the number of new calls that arrive during the experiment,
- $S_1,\ldots,S_N$, the arrival times in seconds of the N new calls,
- $Y_1,\ldots,Y_N$, the call durations in seconds of each of the N new calls.

Some thought will show that samples of each of these random variables, by indicating when every call starts and ends, correspond to one sample function $m(t,s)$. Keep in mind that although these random variables completely specify $m(t,s)$, there are other sets of random variables that also specify $m(t,s)$. For example, instead of referring to the duration of each call, we could instead refer to the time at which each call ends. This yields a different but equivalent set of random variables corresponding to the sample function $m(t,s)$. This example emphasizes that stochastic processes can be quite complex in that each sample function $m(t,s)$ is related to a large number of random variables, each with its own probability model. A complete model of the entire process, $M(t)$, is the model (joint probability mass function or joint probability density function) of all of the individual random variables.

Figure 13.3



Sample functions of four kinds of stochastic processes. $X_{cc}(t)$ is a continuous-time, continuous-value process. $X_{dc}(t)$ is discrete-time, continuous-value process obtained by sampling X_{cc}) every 0.1 seconds. Rounding $X_{cc}(t)$ to the nearest integer yields $X_{cd}(t)$, a continuous-time, discrete-value process. Lastly, $X_{dd}(t)$, a discrete-time, discrete-value process, can be obtained either by sampling $X_{cd}(t)$ or by rounding $X_{dc}(t)$.

Definition 13.4 Discrete-Value and Continuous-Value Processes

$X(t)$ *is a* discrete-value process *if the set of all possible values of* $X(t)$ *at all times* t *is a countable set* S_X*; otherwise* $X(t)$ *is a* continuous-value process.

Definition 13.5 Discrete-Time and Continuous-Time Processes

The stochastic process $X(t)$ *is a* discrete-time process *if* $X(t)$ *is defined only for a set of time instants,* $t_n = nT$*, where* T *is a constant and* n *is an integer; otherwise* $X(t)$ *is a* continuous-time process.

Definition 13.6 Random Sequence

A random sequence X_n *is an ordered sequence of random variables*

$$X_0, X_1, \ldots$$

Quiz 13.1

For the temperature measurements of Example 13.3, construct examples of the measurement process such that the process is

(a) discrete-time, discrete-value,

(b) discrete-time, continuous-value,

(c) continuous-time, discrete-value,

(d) continuous-time, continuous-value.

Quiz 13.1 Solution

(a) We obtain a continuous-time, continuous-value process when we record the temperature as a continuous waveform over time.

(b) If at every moment in time, we round the temperature to the nearest degree, then we obtain a continuous-time, discrete-value process.

(c) If we sample the process in part (a) every T seconds, then we obtain a discrete-time, continuous-value process.

(d) Rounding the samples in part (c) to the nearest integer degree yields a discrete-time, discrete-value process.

Section 13.2

Random Variables from Random Processes

Random Variables from Processes

- Suppose we observe a stochastic process at a particular time instant t_1. In this case, each time we perform the experiment, we observe a sample function $x(t, s)$ and that sample function specifies the value of $x(t_1, s)$. Each time we perform the experiment, we have a new s and we observe a new $x(t_1, s)$. Therefore, each $x(t_1, s)$ is a sample value of a random variable. We use the notation $X(t_1)$ for this random variable. Like any other random variable, it has either a PDF $f_{X(t_1)}(x)$ or a PMF $P_{X(t_1)}(x)$.

- Note that the notation $X(t)$ can refer to either the random process or the random variable that corresponds to the value of the random process at time t.

Example 13.7 Problem

In Example 13.5 of repeatedly rolling a die, what is the PMF of $X(3.5)$?

Example 13.7 Solution

The random variable $X(3.5)$ is the value of the die roll at time 3. In this case,

$$P_{X(3.5)}(x) = \begin{cases} 1/6 & x = 1, \ldots, 6, \\ 0 & \text{otherwise.} \end{cases} \tag{13.3}$$

Example 13.8 Problem

Let $X(t) = R|\cos 2\pi f t|$ be a rectified cosine signal having a random amplitude R with the exponential PDF

$$f_R(r) = \begin{cases} \frac{1}{10}e^{-r/10} & r \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{13.4}$$

What is the PDF $f_{X(t)}(x)$?

Example 13.8 Solution

Since $X(t) \geq 0$ for all t, $\mathrm{P}[X(t) \leq x] = 0$ for $x < 0$. If $x \geq 0$, and $\cos 2\pi ft > 0$,

$$\begin{aligned} \mathrm{P}[X(t) \leq x] &= \mathrm{P}[R \leq x/|\cos 2\pi ft|] \\ &= \int_0^{x/|\cos 2\pi ft|} f_R(r)\, dr = 1 - e^{-x/10|\cos 2\pi ft|}. \end{aligned} \tag{13.5}$$

When $\cos 2\pi ft \neq 0$, the complete CDF of $X(t)$ is

$$F_{X(t)}(x) = \begin{cases} 0 & x < 0, \\ 1 - e^{-x/10|\cos 2\pi ft|} & x \geq 0. \end{cases} \tag{13.6}$$

When $\cos 2\pi ft \neq 0$, the PDF of $X(t)$ is

$$f_{X(t)}(x) = \frac{dF_{X(t)}(x)}{dx} = \begin{cases} \frac{1}{10|\cos 2\pi ft|} e^{-x/10|\cos 2\pi ft|} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{13.7}$$

When $\cos 2\pi ft = 0$ corresponding to $t = \pi/2 + k\pi$, $X(t) = 0$ no matter how large R may be. In this case, $f_{X(t)}(x) = \delta(x)$. In this example, there is a different random variable for each value of t.

Random Vectors from Random Processes

- To answer questions about the random process $X(t)$, we must be able to answer questions about any random vector $\mathbf{X} = \begin{bmatrix} X(t_1) & \cdots & X(t_k) \end{bmatrix}'$ *for any value of* k *and any set of time instants* $t_1, \ldots, t_k$.

- In Section 8.1, the random vector is described by the joint PMF $P_{\mathbf{X}}(\mathbf{x})$ for a discrete-value process $X(t)$ or by the joint PDF $f_{\mathbf{X}}(\mathbf{x})$ for a continuous-value process.

- For a random variable X, we could describe X by its PDF $f_X(x)$, without specifying the exact underlying experiment.

- In the same way, knowledge of the joint PDF $f_{X(t_1),\ldots,X(t_k)}(x_1, \ldots, x_k)$ for all k will allow us to describe a random process without reference to an underlying experiment.

- This is convenient because many experiments lead to the same stochastic process.

Quiz 13.2

In a production line for 1000 Ω resistors, the actual resistance in ohms of each resistor is a uniform $(950, 1050)$ random variable R. The resistances of different resistors are independent. The resistor company has an order for 1% resistors with a resistance between 990 Ω and 1010 Ω. An automatic tester takes one resistor per second and measures its exact resistance. (This test takes one second.) The random process $N(t)$ denotes the number of 1% resistors found in t seconds. The random variable T_r seconds is the elapsed time at which r 1% resistors are found.

(a) What is p, the probability that any single resistor is a 1% resistor?
(b) What is the PMF of $N(t)$?
(c) What is $\mathsf{E}[T_1]$ seconds, the expected time to find the first 1% resistor?
(d) What is the probability that the first 1% resistor is found in exactly 5 seconds?
(e) If the automatic tester finds the first 1% resistor in 10 seconds, find the conditional expected value $\mathsf{E}[T_2|T_1 = 10]$ of the time of finding the second 1% resistor?

Quiz 13.2 Solution

(a) Each resistor has resistance R in ohms with uniform PDF

$$f_R(r) = \begin{cases} 0.01 & 950 \leq r \leq 1050 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The probability that a test produces a 1% resistor is

$$p = \mathrm{P}[990 \leq R \leq 1010] = \int_{990}^{1010} (0.01)\, dr = 0.2. \tag{2}$$

(b) In t seconds, exactly t resistors are tested. Each resistor is a 1% resistor with probability $p = 0.2$, independent of any other resistor. Consequently, the number of 1% resistors found has the binomial $(t, 0.2)$ PMF

$$P_{N(t)}(n) = \binom{t}{n} (0.2)^n (0.8)^{t-n}. \tag{3}$$

(c) First we will find the PMF of T_1. This problem is easy if we view each resistor test as an independent trial. A success occurs on a trial with probability $p = 0.2$ if we find a 1% resistor. The first 1% resistor is found at time $T_1 = t$ if we observe failures on trials $1, \ldots, t-1$ followed by a success on trial t.

[Continued]

Quiz 13.2 Solution (Continued 2)

Hence, just as in Example 3.8, T_1 has the geometric (0.2) PMF

$$P_{T_1}(t) = \begin{cases} (0.8)^{t-1}(0.2) & t = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

From Theorem 3.5, a geometric random variable with success probability p has expected value $1/p$. In this problem, $E[T_1] = 1/p = 5$.

(a) Since $p = 0.2$, the probability the first 1% resistor is found in exactly five seconds is $P_{T_1}(5) = (0.8)^4(0.2) = 0.08192$.

(b) Note that once we find the first 1% resistor, the number of additional trials needed to find the second 1% resistor once again has a geometric PMF with expected value $1/p$ since each independent trial is a success with probability p. That is, $T_2 = T_1 + T'$ where T' is independent and identically distributed to T_1. Thus

$$\begin{aligned} E[T_2|T_1 = 10] &= E[T_1|T_1 = 10] + E\left[T'|T_1 = 10\right] \\ &= 10 + E\left[T'\right] = 10 + 5 = 15. \end{aligned} \tag{5}$$

Section 13.3

Independent, Identically Distributed Random Sequences

Example 13.9

In Quiz 13.2, each independent resistor test required exactly 1 second. Let R_n equal the number of 1% resistors found during minute n. The random variable R_n has the binomial PMF

$$P_{R_n}(r) = \binom{60}{r} p^r (1-p)^{60-r}. \tag{13.8}$$

Since each resistor is a 1% resistor independent of all other resistors, the number of 1% resistors found in each minute is independent of the number found in other minutes. Thus $R_1, R_2, \ldots$ is an iid random sequence.

Theorem 13.1

Let X_n denote an iid random sequence. For a discrete-value process, the sample vector $\mathbf{X} = \begin{bmatrix} X_{n_1} & \cdots & X_{n_k} \end{bmatrix}'$ has joint PMF

$$P_{\mathbf{X}}(\mathbf{x}) = P_X(x_1)\, P_X(x_2) \cdots P_X(x_k) = \prod_{i=1}^{k} P_X(x_i)\,.$$

. . . .

For a continuous-value process, the joint PDF of $\mathbf{X} = \begin{bmatrix} X_{n_1} & \cdots & , X_{n_k} \end{bmatrix}'$ is

$$f_{\mathbf{X}}(\mathbf{x}) = f_X(x_1)\, f_X(x_2) \cdots f_X(x_k) = \prod_{i=1}^{k} f_X(x_i)\,.$$

Definition 13.7 Bernoulli Process

A Bernoulli (p) *process* X_n *is an iid random sequence in which each* X_n *is a Bernoulli* (p) *random variable.*

Example 13.10

In a common model for communications, the output $X_1, X_2, \ldots$ of a binary source is modeled as a Bernoulli ($p = 1/2$) process.

Example 13.11 Problem

For a Bernoulli (p) process X_n, find the joint PMF of $\mathbf{X} = \begin{bmatrix} X_1 & \cdots & X_n \end{bmatrix}'$.

Example 13.11 Solution

For a single sample X_i, we can write the Bernoulli PMF in the following way:

$$P_{X_i}(x_i) = \begin{cases} p^{x_i}(1-p)^{1-x_i} & x_i \in \{0,1\}, \\ 0 & \text{otherwise.} \end{cases} \tag{13.9}$$

When $x_i \in \{0,1\}$ for $i = 1, \ldots, n$, the joint PMF can be written as

$$P_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^k(1-p)^{n-k}, \tag{13.10}$$

where $k = x_1 + \cdots + x_n$. The complete expression for the joint PMF is

$$P_{\mathbf{X}}(\mathbf{x}) = \begin{cases} p^{x_1+\cdots+x_n}(1-p)^{n-(x_1+\cdots+x_n)} & x_i \in \{0,1\}, i = 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases} \tag{13.11}$$

Quiz 13.3

For an iid random sequence X_n of Gaussian $(0, 1)$ random variables, find the joint PDF of $\mathbf{X} = \begin{bmatrix} X_1 & \cdots & X_m \end{bmatrix}'$.

Quiz 13.3 Solution

Since each X_i is a $N(0,1)$ random variable, each X_i has PDF

$$f_{X_i)}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \qquad (1)$$

By Theorem 13.1, the joint PDF of $\mathbf{X} = \begin{bmatrix} X_1 & \cdots X_n \end{bmatrix}'$ is

$$f_{\mathbf{X}}(\mathbf{x}) = f_{X(1),\ldots,X(n)}(x_1, \ldots, x_n) = \prod_{i=1}^{k} f_X(x_i) = \frac{1}{(2\pi)^{n/2}} e^{-(x_1^2+\cdots+x_n^2)/2}. \qquad (2)$$

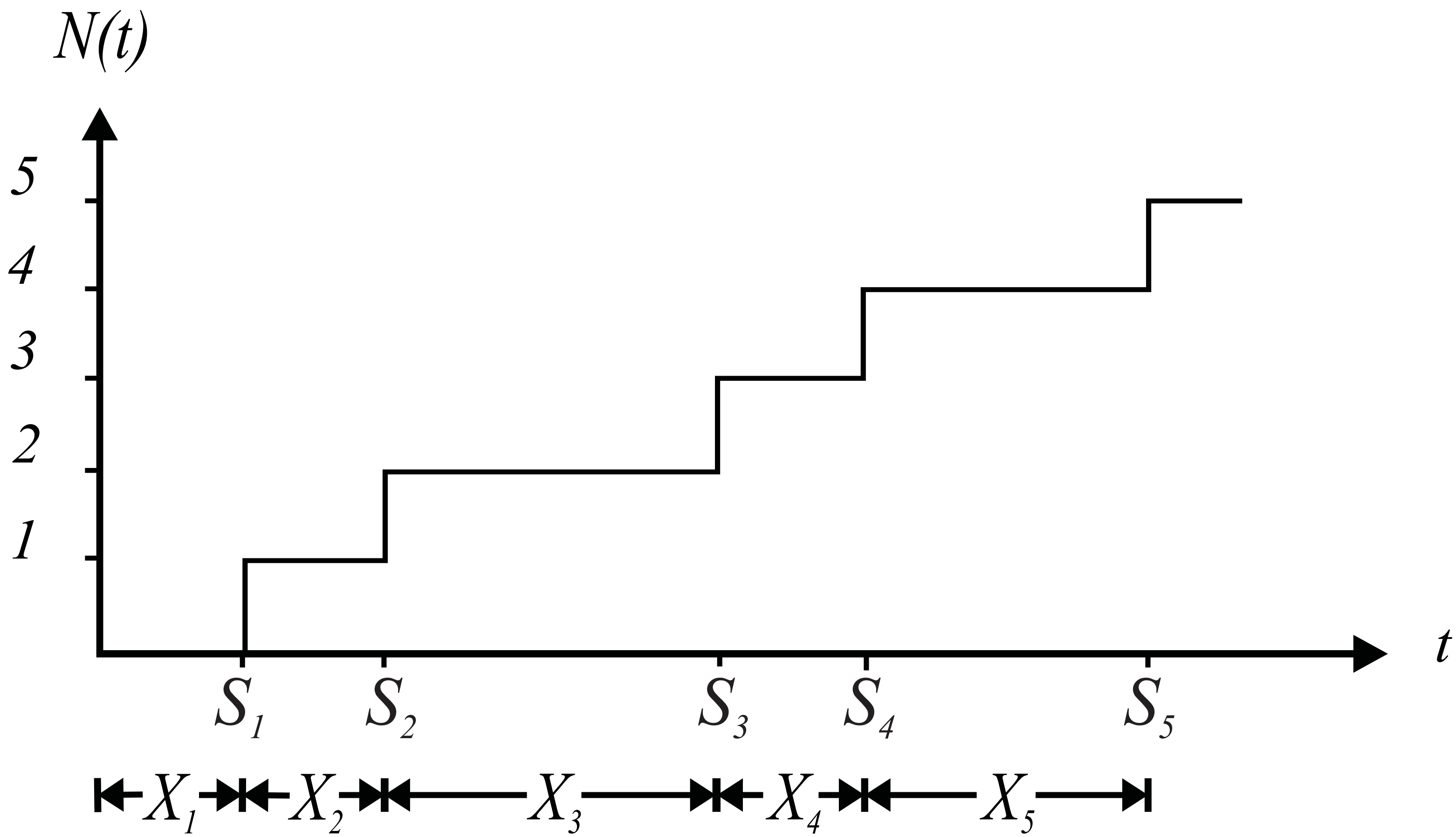
Section 13.4

The Poisson Process

Definition 13.8 Counting Process

A stochastic process $N(t)$ *is a* counting process *if for every sample function,* $n(t,s) = 0$ *for* $t < 0$ *and* $n(t,s)$ *is integer-valued and nondecreasing with time.*

Figure 13.4



Sample path of a counting process.

Definition 13.9 Poisson Process

A counting process $N(t)$ *is a* Poisson process *of rate* λ *if*

(a) The number of arrivals in any interval $(t_0, t_1]$, $N(t_1) - N(t_0)$, *is a Poisson random variable with expected value* $\lambda(t_1 - t_0)$.

(b) For any pair of nonoverlapping intervals $(t_0, t_1]$ *and* $(t'_0, t'_1]$, *the number of arrivals in each interval,* $N(t_1) - N(t_0)$ *and* $N(t'_1) - N(t'_0)$, *respectively, are independent random variables.*

Theorem 13.2

For a Poisson process $N(t)$ of rate λ, the joint PMF of

$$\mathbf{N} = \left[N(t_1), \ldots, N(t_k)\right]'$$

for ordered time instances $t_1 < \cdots < t_k$ is

$$P_{\mathbf{N}}(\mathbf{n}) = \begin{cases} \frac{\alpha_1^{n_1} e^{-\alpha_1}}{n_1!} \frac{\alpha_2^{n_2-n_1} e^{-\alpha_2}}{(n_2-n_1)!} \cdots \frac{\alpha_k^{n_k-n_{k-1}} e^{-\alpha_k}}{(n_k-n_{k-1})!} & 0 \leq n_1 \leq \cdots \leq n_k, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha_1 = \lambda t_1$, and for $i = 2, \ldots, k$, $\alpha_i = \lambda(t_i - t_{i-1})$.

Proof: Theorem 13.2

Let $M_1 = N(t_1)$ and for $i > 1$, let $M_i = N(t_i) - N(t_{i-1})$. By the definition of the Poisson process, $M_1, \ldots, M_k$ is a collection of independent Poisson random variables such that $\mathrm{E}[M_i] = \alpha_i$.

$$\begin{aligned} P_{\mathbf{N}}(\mathbf{n}) &= P_{M_1,M_2,\ldots,M_k}(n_1, n_2 - n_1, \ldots, n_k - n_{k-1}) \\ &= P_{M_1}(n_1)\, P_{M_2}(n_2 - n_1) \cdots P_{M_k}(n_k - n_{k-1})\,. \end{aligned} \tag{13.15}$$

The theorem follows by substituting Equation (13.14) for $P_{M_i}(n_i - n_{i-1})$.

Theorem 13.3

For a Poisson process of rate λ, the interarrival times $X_1, X_2, \ldots$ are an iid random sequence with the exponential PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Theorem 13.3

Given $X_1 = x_1, X_2 = x_2, \ldots, X_{n-1} = x_{n-1}$, arrival $n-1$ occurs at time

$$t_{n-1} = x_1 + \cdots + x_{n-1}. \tag{13.16}$$

For $x > 0$, $X_n > x$ if and only if there are no arrivals in the interval $(t_{n-1}, t_{n-1} + x]$. The number of arrivals in $(t_{n-1}, t_{n-1} + x]$ is independent of the past history described by $X_1, \ldots, X_{n-1}$. This implies

$$\begin{aligned} \text{P}\left[X_n > x | X_1 = x_1, \ldots, X_{n-1} = x_{n-1}\right] &= \text{P}\left[N(t_{n-1} + x) - N(t_{n-1}) = 0\right] \\ &= e^{-\lambda x}. \end{aligned} \tag{13.17}$$

Thus X_n is independent of $X_1, \ldots, X_{n-1}$ and has the exponential CDF

$$F_{X_n}(x) = 1 - \text{P}\left[X_n > x\right] = \begin{cases} 1 - e^{-\lambda x} & x > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{13.18}$$

From the derivative of the CDF, we see that X_n has the exponential PDF $f_{X_n}(x) = f_X(x)$ in the statement of the theorem.

Theorem 13.4

A counting process with independent exponential (λ) interarrivals $X_1, X_2, \ldots$ is a Poisson process of rate λ.

Quiz 13.4

Data packets transmitted by a modem over a phone line form a Poisson process of rate 10 packets/sec. Using M_k to denote the number of packets transmitted in the kth hour, find the joint PMF of M_1 and M_2.

Quiz 13.4 Solution

The first and second hours are nonoverlapping intervals. Since one hour equals 3600 sec and the Poisson process has a rate of 10 packets/sec, the expected number of packets in each hour is $\mathrm{E}[M_i] = \alpha = 36,000$. This implies M_1 and M_2 are independent Poisson random variables each with PMF

$$P_{M_i}(m) = \begin{cases} \frac{\alpha^m e^{-\alpha}}{m!} & m = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Since M_1 and M_2 are independent, the joint PMF of M_1 and M_2 is

$$P_{M_1,M_2}(m_1, m_2) = P_{M_1}(m_1)\, P_{M_2}(m_2) = \begin{cases} \frac{\alpha^{m_1+m_2} e^{-2\alpha}}{m_1! m_2!} & m_1 = 0, 1, \ldots; \\ & m_2 = 0, 1, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Section 13.5

Properties of the Poisson Process

Theorem 13.5

Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes of rates λ_1 and λ_2. The counting process $N(t) = N_1(t) + N_2(t)$ is a Poisson process of rate $\lambda_1 + \lambda_2$.

Proof: Theorem 13.5

We show that the interarrival times of the $N(t)$ process are iid exponential random variables. Suppose the $N(t)$ process just had an arrival. Whether that arrival was from $N_1(t)$ or $N_2(t)$, X_i, the residual time until the next arrival of $N_i(t)$, has an exponential PDF since $N_i(t)$ is a memoryless process. Further, X, the next interarrival time of the $N(t)$ process, can be written as $X = \min(X_1, X_2)$. Since X_1 and X_2 are independent of the past interarrival times, X must be independent of the past interarrival times. In addition, we observe that $X > x$ if and only if $X_1 > x$ and $X_2 > x$. This implies $P[X > x] = P[X_1 > x, X_2 > x]$. Since $N_1(t)$ and $N_2(t)$ are independent processes, X_1 and X_2 are independent random variables so that

$$P\left[X > x\right] = P\left[X_1 > x\right] P\left[X_2 > x\right] = \begin{cases} 1 & x < 0, \\ e^{-(\lambda_1+\lambda_2)x} & x \geq 0. \end{cases} \tag{13.20}$$

Thus X is an exponential $(\lambda_1 + \lambda_2)$ random variable.

Example 13.12 Problem

Cars, trucks, and buses arrive at a toll booth as independent Poisson processes with rates $\lambda_c = 1.2$ cars/minute, $\lambda_t = 0.9$ trucks/minute, and $\lambda_b = 0.7$ buses/minute. In a 10-minute interval, what is the PMF of N, the number of vehicles (cars, trucks, or buses) that arrive?

Example 13.12 Solution

By Theorem 13.5, the arrival of vehicles is a Poisson process of rate $\lambda = 1.2 + 0.9 + 0.7 = 2.8$ vehicles per minute. In a 10-minute interval, $\lambda T = 28$ and N has PMF

$$P_N(n) = \begin{cases} 28^n e^{-28}/n! & n = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{13.21}$$

Theorem 13.6

The counting processes $N_1(t)$ and $N_2(t)$ derived from a Bernoulli decomposition of the Poisson process $N(t)$ are independent Poisson processes with rates λp and $\lambda(1-p)$.

Proof: Theorem 13.6

Let $X_1^{(i)}, X_2^{(i)}, \ldots$ denote the interarrival times of the process $N_i(t)$. We will verify that $X_1^{(1)}, X_2^{(1)}, \ldots$ and $X_1^{(2)}, X_2^{(2)}, \ldots$ are independent random sequences, each with exponential CDFs. We first consider the interarrival times of the $N_1(t)$ process. Suppose time t marked arrival $n-1$ of the $N_1(t)$ process. The next interarrival time $X_n^{(1)}$ depends only on future coin flips and future arrivals of the memoryless $N(t)$ process and thus is independent of all past interarrival times of either the $N_1(t)$ or $N_2(t)$ processes. This implies the $N_1(t)$ process is independent of the $N_2(t)$ process. All that remains is to show that $X_n^{(1)}$ is an exponential random variable. We observe that $X_n^{(1)} > x$ if there are no type 1 arrivals in the interval $[t, t+x]$. For the interval $[t, t+x]$, let N_1 and N denote the number of arrivals of the $N_1(t)$ and $N(t)$ processes. In terms of N_1 and N, we can write

$$\mathrm{P}\left[X_n^{(1)} > x\right] = P_{N_1}(0) = \sum_{n=0}^{\infty} P_{N_1|N}(0|n)\, P_N(n)\,. \tag{13.22}$$

[Continued]

Proof: Theorem 13.6 (Continued 2)

Given $N = n$ total arrivals, $N_1 = 0$ if each of these arrivals is labeled type 2. This will occur with probability $P_{N_1|N}(0|n) = (1-p)^n$. Thus

$$\mathrm{P}\left[X_n^{(1)} > x\right] = \sum_{n=0}^{\infty} (1-p)^n \frac{(\lambda x)^n e^{-\lambda x}}{n!} = e^{-p\lambda x} \underbrace{\sum_{n=0}^{\infty} \frac{[(1-p)\lambda x]^n e^{-(1-p)\lambda x}}{n!}}_{1}. \tag{13.23}$$

Thus $\mathrm{P}[X_n^{(1)} > x] = e^{-p\lambda x}$; each $X_n^{(1)}$ has an exponential PDF with mean $1/(p\lambda)$. It follows that $N_1(t)$ is a Poisson process of rate $\lambda_1 = p\lambda$. The same argument can be used to show that each $X_n^{(2)}$ has an exponential PDF with mean $1/[(1-p)\lambda]$, implying $N_2(t)$ is a Poisson process of rate $\lambda_2 = (1-p)\lambda$.

Example 13.13 Problem

A corporate Web server records hits (requests for HTML documents) as a Poisson process at a rate of 10 hits per second. Each page is either an internal request (with probability 0.7) from the corporate intranet or an external request (with probability 0.3) from the Internet. Over a 10-minute interval, what is the joint PMF of I, the number of internal requests, and X, the number of external requests?

Example 13.13 Solution

By Theorem 13.6, the internal and external request arrivals are independent Poisson processes with rates of 7 and 3 hits per second. In a 10-minute (600-second) interval, I and X are independent Poisson random variables with parameters $\alpha_I = 7(600) = 4200$ and $\alpha_X = 3(600) = 1800$ hits. The joint PMF of I and X is

$$P_{I,X}(i,x) = P_I(i)\, P_X(x) = \begin{cases} \frac{(4200)^i e^{-4200}}{i!} \frac{(1800)^x e^{-1800}}{x!} & i, x \in \{0, 1, \ldots\}, \\ 0 & \text{otherwise.} \end{cases} \quad (13.24)$$

Theorem 13.7

Let $N(t) = N_1(t) + N_2(t)$ be the sum of two independent Poisson processes with rates λ_1 and λ_2. Given that the $N(t)$ process has an arrival, the conditional probability that the arrival is from $N_1(t)$ is $\lambda_1/(\lambda_1 + \lambda_2)$.

Proof: Theorem 13.7

We can view $N_1(t)$ and $N_2(t)$ as being derived from a Bernoulli decomposition of $N(t)$ in which an arrival of $N(t)$ is labeled a type 1 arrival with probability $\lambda_1/(\lambda_1 + \lambda_2)$. By Theorem 13.6, $N_1(t)$ and $N_2(t)$ are independent Poisson processes with rate λ_1 and λ_2, respectively. Moreover, given an arrival of the $N(t)$ process, the conditional probability that an arrival is an arrival of the $N_1(t)$ process is also $\lambda_1/(\lambda_1 + \lambda_2)$.

Quiz 13.5

Let $N(t)$ be a Poisson process of rate λ. Let $N'(t)$ be a process in which we count only even-numbered arrivals; that is, arrivals $2, 4, 6, \ldots$, of the process $N(t)$. Is $N'(t)$ a Poisson process?

Quiz 13.5 Solution

To answer whether $N'(t)$ is a Poisson process, we look at the interarrival times. Let $X_1, X_2, \ldots$ denote the interarrival times of the $N(t)$ process. Since we count only even-numbered arrival for $N'(t)$, the time until the first arrival of the $N'(t)$ is $Y_1 = X_1 + X_2$. Since X_1 and X_2 are independent exponential (λ) random variables, Y_1 is an Erlang $(n = 2, \lambda)$ random variable; see Theorem 9.9. Since $Y_i(t)$, the ith interarrival time of the $N'(t)$ process, has the same PDF as $Y_1(t)$, we can conclude that the interarrival times of $N'(t)$ are not exponential random variables. Thus $N'(t)$ is *not* a Poisson process.

Section 13.6

The Brownian Motion Process

Definition 13.10 Brownian Motion Process

A Brownian motion process $W(t)$ *has the property that* $W(0) = 0$, *and for* $\tau > 0$, $W(t+\tau) - W(t)$ *is a Gaussian* $(0, \sqrt{\alpha\tau})$ *random variable that is independent of* $W(t')$ *for all* $t' \leq t$.

Theorem 13.8

For the Brownian motion process $W(t)$, the PDF of

$$\mathbf{W} = \left[W(t_1), \ldots, W(t_k)\right]'$$

is

$$f_{\mathbf{W}}(\mathbf{w}) = \prod_{n=1}^{k} \frac{1}{\sqrt{2\pi\alpha(t_n - t_{n-1})}} e^{-(w_n - w_{n-1})^2/[2\alpha(t_n - t_{n-1})]}.$$

Proof: Theorem 13.8

Since $W(0) = 0$, $W(t_1) = X(t_1) - W(0)$ is a Gaussian random variable. Given time instants $t_1, \ldots, t_k$, we define $t_0 = 0$ and, for $n = 1, \ldots, k$, we can define the increments $X_n = W(t_n) - W(t_{n-1})$. Note that $X_1, \ldots, X_k$ are independent random variables such that X_n is Gaussian $(0, \sqrt{\alpha(t_n - t_{n-1})})$.

$$f_{X_n}(x) = \frac{1}{\sqrt{2\pi\alpha(t_n - t_{n-1})}} e^{-x^2/[2\alpha(t_n - t_{n-1})]}. \tag{13.26}$$

Note that $\mathbf{W} = \mathbf{w}$ if and only if $W_1 = w_1$ and for $n = 2, \ldots, k$, $X_n = w_n - w_{n-1}$. Although we omit some significant steps that can be found in Problem 13.6.5, this does imply

$$f_{\mathbf{W}}(\mathbf{w}) = \prod_{n=1}^{k} f_{X_n}(w_n - w_{n-1}). \tag{13.27}$$

The theorem follows from substitution of Equation (13.26) into Equation (13.27).

Quiz 13.6

Let $W(t)$ be a Brownian motion process with variance $\text{Var}[W(t)] = \alpha t$. Show that $X(t) = W(t)/\sqrt{\alpha}$ is a Brownian motion process with variance $\text{Var}[X(t)] = t$.

Quiz 13.6 Solution

First, we note that for $t > s$,

$$X(t) - X(s) = \frac{W(t) - W(s)}{\sqrt{\alpha}}. \tag{1}$$

Since $W(t) - W(s)$ is a Gaussian random variable, Theorem 4.13 states that $W(t) - W(s)$ is Gaussian with expected value

$$\mathsf{E}[X(t) - X(s)] = \frac{\mathsf{E}[W(t) - W(s)]}{\sqrt{\alpha}} = 0 \tag{2}$$

and variance

$$\mathsf{E}\left[(W(t) - W(s))^2\right] = \frac{\mathsf{E}\left[(W(t) - W(s))^2\right]}{\alpha} = \frac{\alpha(t-s)}{\alpha}. \tag{3}$$

Consider $s' \leq s < t$. Since $s \geq s'$, $W(t) - W(s)$ is independent of $W(s')$. This implies $[W(t) - W(s)]/\sqrt{\alpha}$ is independent of $W(s')/\sqrt{\alpha}$ for all $s \geq s'$. That is, $X(t) - X(s)$ is independent of $X(s')$ for all $s \geq s'$. Thus $X(t)$ is a Brownian motion process with variance $\text{Var}[X(t)] = t$.

Section 13.7

Expected Value and Correlation

Definition 13.11 The Expected Value of a Process

The expected value *of a stochastic process $X(t)$ is the deterministic function*

$$\mu_X(t) = \mathsf{E}[X(t)].$$

Definition 13.12 Autocovariance

The autocovariance *function of the stochastic process* $X(t)$ *is*

$$C_X(t,\tau) = \mathsf{Cov}\left[X(t), X(t+\tau)\right].$$

. . . .

The autocovariance *function of the random sequence* X_n *is*

$$C_X[m,k] = \mathsf{Cov}\left[X_m, X_{m+k}\right].$$

Definition 13.13 Autocorrelation Function

The autocorrelation function *of the stochastic process* $X(t)$ *is*

$$R_X(t,\tau) = \mathsf{E}\left[X(t)X(t+\tau)\right].$$

. . . .

The autocorrelation *function of the random sequence* X_n *is*

$$R_X[m,k] = \mathsf{E}\left[X_m X_{m+k}\right].$$

Theorem 13.9

The autocorrelation and autocovariance functions of a process $X(t)$ satisfy

$$C_X(t,\tau) = R_X(t,\tau) - \mu_X(t)\mu_X(t+\tau).$$

. . . .

The autocorrelation and autocovariance functions of a random sequence X_n satisfy

$$C_X[n,k] = R_X[n,k] - \mu_X(n)\mu_X(n+k).$$

Example 13.14 Problem

Find the autocovariance $C_X(t\,\tau)$ and autocorrelation $R_X(t,\tau)$ of the Brownian motion process $X(t)$.

Example 13.14 Solution

From the definition of the Brownian motion process, we know that $\mu_X(t) = 0$. Thus the autocorrelation and autocovariance are equal: $C_X(t,\tau) = R_X(t,\tau)$. To find the autocorrelation $R_X(t,\tau)$, we exploit the independent increments property of Brownian motion. For the moment, we assume $\tau \geq 0$ so we can write $R_X(t,\tau) = \mathsf{E}[X(t)X(t+\tau)]$. Because the definition of Brownian motion refers to $X(t+\tau) - X(t)$, we introduce this quantity by substituting $X(t+\tau) = X(t+\tau) - X(t) + X(t)$. The result is

$$\begin{aligned} R_X(t,\tau) &= \mathsf{E}\left[X(t)[(X(t+\tau) - X(t)) + X(t)]\right] \\ &= \mathsf{E}\left[X(t)[X(t+\tau) - X(t)]\right] + \mathsf{E}\left[X^2(t)\right]. \end{aligned} \tag{13.28}$$

By the definition of Brownian motion, $X(t)$ and $X(t+\tau) - X(t)$ are independent, with zero expected value. This implies

$$\mathsf{E}\left[X(t)[X(t+\tau) - X(t)]\right] = \mathsf{E}\left[X(t)\right]\mathsf{E}\left[X(t+\tau) - X(t)\right] = 0. \tag{13.29}$$

Furthermore, since $\mathsf{E}[X(t)] = 0$, $\mathsf{E}[X^2(t)] = \mathsf{Var}[X(t)]$. Therefore, Equation (13.28) implies

$$R_X(t,\tau) = \mathsf{E}\left[X^2(t)\right] = \alpha t, \qquad \tau \geq 0. \tag{13.30}$$

When $\tau < 0$, we can reverse the labels in the preceding argument to show that $R_X(t,\tau) = \alpha(t+\tau)$. For arbitrary t and τ we can combine these statements to write

$$R_X(t,\tau) = \alpha \min\{t, t+\tau\}. \tag{13.31}$$

Quiz 13.7

$X(t)$ has expected value $\mu_X(t)$ and autocorrelation $R_X(t,\tau)$. We make the noisy observation $Y(t) = X(t) + N(t)$, where $N(t)$ is a random noise process independent of $X(t)$ with $\mu_N(t) = 0$ and autocorrelation $R_N(t,\tau)$. Find the expected value and autocorrelation of $Y(t)$.

Quiz 13.7 Solution

First we find the expected value

$$\mu_Y(t) = \mu_X(t) + \mu_N(t) = \mu_X(t). \tag{1}$$

To find the autocorrelation, we observe that since $X(t)$ and $N(t)$ are independent and since $N(t)$ has zero expected value,

$$\mathsf{E}[X(t)N(t')] = \mathsf{E}[X(t)]\,\mathsf{E}[N(t')] = 0.$$

Since $R_Y(t,\tau) = \mathsf{E}[Y(t)Y(t+\tau)]$, we have

$$\begin{aligned} R_Y(t,\tau) &= \mathsf{E}\left[(X(t)+N(t))\,(X(t+\tau)+N(t+\tau))\right] \\ &= \mathsf{E}\left[X(t)X(t+\tau)\right] + \mathsf{E}\left[X(t)N(t+\tau)\right] \\ &\qquad + \mathsf{E}\left[X(t+\tau)N(t)\right] + \mathsf{E}\left[N(t)N(t+\tau)\right] \\ &= R_X(t,\tau) + R_N(t,\tau). \end{aligned} \tag{2}$$

Section 13.8

Stationary Processes

Definition 13.14 Stationary Process

A stochastic process $X(t)$ *is* stationary *if and only if for all sets of time instants* $t_1, \ldots, t_m$, *and any time difference* τ,

$$f_{X(t_1),\ldots,X(t_m)}(x_1,\ldots,x_m) = f_{X(t_1+\tau),\ldots,X(t_m+\tau)}(x_1,\ldots,x_m).$$

. . . .

A random sequence X_n *is* stationary *if and only if for any set of integer time instants* $n_1, \ldots, n_m$, *and integer time difference* k,

$$f_{X_{n_1},\ldots,X_{n_m}}(x_1,\ldots,x_m) = f_{X_{n_1+k},\ldots,X_{n_m+k}}(x_1,\ldots,x_m).$$

Example 13.15 Problem

Is the Brownian motion process with parameter α introduced in Section 13.6 stationary?

Example 13.15 Solution

For Brownian motion, $X(t_1)$ is the Gaussian $(0, \sqrt{\alpha\tau_1})$ random variable. Similarly, $X(t_2)$ is Gaussian $(0, \sqrt{\alpha\tau_2})$. Since $X(t_1)$ and $X(t_2)$ do not have the same variance, $f_{X(t_1)}(x) \neq f_{X(t_2)}(x)$, and the Brownian motion process is not stationary.

Theorem 13.10

Let $X(t)$ be a stationary random process. For constants $a > 0$ and b, $Y(t) = aX(t) + b$ is also a stationary process.

Proof: Theorem 13.10

For an arbitrary set of time samples $t_1, \ldots, t_n$, we need to find the joint PDF of $Y(t_1), \ldots, Y(t_n)$. We have solved this problem in Theorem 8.5 where we found that

$$f_{Y(t_1),\ldots,Y(t_n)}(y_1, \ldots, y_n) = \frac{1}{|a|^n} f_{X(t_1),\ldots,X(t_n)}\left(\frac{y_1 - b}{a}, \ldots, \frac{y_n - b}{a}\right). \tag{13.33}$$

Since the process $X(t)$ is stationary, we can write

$$\begin{aligned} f_{Y(t_1+\tau),\ldots,Y(t_n+\tau)}(y_1, \ldots, y_n) &= \frac{1}{a^n} f_{X(t_1+\tau),\ldots,X(t_n+\tau)}\left(\frac{y_1 - b}{a}, \ldots, \frac{y_n - b}{a}\right) \\ &= \frac{1}{a^n} f_{X(t_1),\ldots,X(t_n)}\left(\frac{y_1 - b}{a}, \ldots, \frac{y_n - b}{a}\right) \\ &= f_{Y(t_1),\ldots,Y(t_n)}(y_1, \ldots, y_n). \end{aligned} \tag{13.34}$$

Thus $Y(t)$ is also a stationary random process.

Theorem 13.11

For a stationary process $X(t)$, the expected value, the autocorrelation, and the autocovariance have the following properties for all t:

(a) $\mu_X(t) = \mu_X$,

(b) $R_X(t, \tau) = R_X(0, \tau) = R_X(\tau)$,

(c) $C_X(t, \tau) = R_X(\tau) - \mu_X^2 = C_X(\tau)$.

. . . .

For a stationary random sequence X_n the expected value, the autocorrelation, and the autocovariance satisfy for all n

(a) $\mathrm{E}[X_n] = \mu_X$,

(b) $R_X[n, k] = R_X[0, k] = R_X[k]$,

(c) $C_X[n, k] = R_X[k] - \mu_X^2 = C_X[k]$.

Proof: Theorem 13.11

By Definition 13.14, stationarity of $X(t)$ implies $f_{X(t)}(x) = f_{X(0)}(x)$, so that

$$\mu_X(t) = \int_{-\infty}^{\infty} x f_{X(t)}(x)\ dx = \int_{-\infty}^{\infty} x f_{X(0)}(x)\ dx = \mu_X(0). \tag{13.35}$$

Note that $\mu_X(0)$ is just a constant that we call μ_X. Also, by Definition 13.14,

$$f_{X(t),X(t+\tau)}(x_1, x_2) = f_{X(t-t),X(t+\tau-t)}(x_1, x_2), \tag{13.36}$$

so that

$$\begin{aligned} R_X(t,\tau) = \mathrm{E}\left[X(t)X(t+\tau)\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(0),X(\tau)}(x_1, x_2)\ dx_1\, dx_2 \\ &= R_X(0,\tau) = R_X(\tau). \end{aligned} \tag{13.37}$$

Lastly, by Theorem 13.9,

$$C_X(t,\tau) = R_X(t,\tau) - \mu_X^2 = R_X(\tau) - \mu_X^2 = C_X(\tau). \tag{13.38}$$

We obtain essentially the same relationships for random sequences by replacing $X(t)$ and $X(t+\tau)$ with X_n and X_{n+k}.

Example 13.16 Problem

At the receiver of an AM radio, the received signal contains a cosine carrier signal at the carrier frequency f_c with a random phase Θ that is a sample value of the uniform $(0, 2\pi)$ random variable. The received carrier signal is

$$X(t) = A\cos(2\pi f_c t + \Theta). \tag{13.39}$$

What are the expected value and autocorrelation of the process $X(t)$?

Example 13.16 Solution

The phase has PDF

$$f_\Theta(\theta) = \begin{cases} 1/(2\pi) & 0 \le \theta \le 2\pi, \\ 0 & \text{otherwise.} \end{cases} \tag{13.40}$$

For any fixed angle α and integer k,

$$\begin{aligned} \mathrm{E}\left[\cos(\alpha + k\Theta)\right] &= \int_0^{2\pi} \cos(\alpha + k\theta) \frac{1}{2\pi}\, d\theta \\ &= \left.\frac{\sin(\alpha + k\theta)}{k}\right|_0^{2\pi} = \frac{\sin(\alpha + k2\pi) - \sin\alpha}{k} = 0. \end{aligned} \tag{13.41}$$

Choosing $\alpha = 2\pi f_c t$, and $k = 1$, $\mathrm{E}[X(t)]$ is

$$\mu_X(t) = \mathrm{E}\left[A\cos(2\pi f_c t + \Theta)\right] = 0. \tag{13.42}$$

We will use the identity $\cos A \cos B = [\cos(A - B) + \cos(A + B)]/2$ to find the autocorrelation:

$$\begin{aligned} R_X(t,\tau) &= \mathrm{E}\left[A\cos(2\pi f_c t + \Theta) A\cos(2\pi f_c(t+\tau) + \Theta)\right] \\ &= \frac{A^2}{2} \mathrm{E}\left[\cos(2\pi f_c \tau) + \cos(2\pi f_c(2t + \tau) + 2\Theta)\right]. \end{aligned} \tag{13.43}$$

[Continued]

Example 13.16 Solution (Continued 2)

For $\alpha = 2\pi f_c(t+\tau)$ and $k = 2$,

$$\mathrm{E}\left[\cos(2\pi f_c(2t+\tau) + 2\Theta)\right] = \mathrm{E}\left[\cos(\alpha + k\Theta)\right] = 0. \tag{13.44}$$

Thus

$$R_X(t,\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau) = R_X(\tau). \tag{13.45}$$

Therefore, $X(t)$ has the properties of a stationary stochastic process listed in Theorem 13.11.

Quiz 13.8

Let $X_1, X_2, \ldots$ be an iid random sequence. Is $X_1, X_2, \ldots$ a stationary random sequence?

Quiz 13.8 Solution

From Definition 13.14, $X_1, X_2, \ldots$ is a stationary random sequence if for all sets of time instants $n_1, \ldots, n_m$ and time offset k,

$$f_{X_{n_1},\ldots,X_{n_m}}(x_1, \ldots, x_m) = f_{X_{n_1+k},\ldots,X_{n_m+k}}(x_1, \ldots, x_m). \tag{1}$$

Since the random sequence is iid,

$$f_{X_{n_1},\ldots,X_{n_m}}(x_1, \ldots, x_m) = f_X(x_1)\, f_X(x_2) \cdots f_X(x_m). \tag{2}$$

Similarly, for time instants $n_1 + k, \ldots, n_m + k$,

$$f_{X_{n_1+k},\ldots,X_{n_m+k}}(x_1, \ldots, x_m) = f_X(x_1)\, f_X(x_2) \cdots f_X(x_m). \tag{3}$$

We can conclude that the iid random sequence is stationary.

Section 13.9

Wide Sense Stationary Stochastic Processes

Definition 13.15 Wide Sense Stationary

$X(t)$ *is a* wide sense stationary stochastic process *if and only if for all* t,

$$\mathsf{E}[X(t)] = \mu_X, \quad \textit{and} \quad R_X(t,\tau) = R_X(0,\tau) = R_X(\tau).$$

. . . .

X_n *is a* wide sense stationary random sequence *if and only if for all* n,

$$\mathsf{E}[X_n] = \mu_X, \quad \textit{and} \quad R_X[n,k] = R_X[0,k] = R_X[k].$$

Example 13.17

In Example 13.16, we observe that $\mu_X(t) = 0$ and

$$R_X(t, \tau) = (A^2/2)\cos 2\pi f_c \tau.$$

Thus the random phase carrier $X(t)$ is a wide sense stationary process.

Theorem 13.12

For a wide sense stationary process $X(t)$, the autocorrelation function $R_X(\tau)$ has the following properties:

$$R_X(0) \geq 0, \qquad R_X(\tau) = R_X(-\tau), \qquad R_X(0) \geq |R_X(\tau)|.$$

. . . .

If X_n is a wide sense stationary random sequence:

$$R_X[0] \geq 0, \qquad R_X[k] = R_X[-k], \qquad R_X[0] \geq |R_X[k]|.$$

Proof: Theorem 13.12

For the first property, $R_X(0) = R_X(t,0) = \mathsf{E}[X^2(t)]$. Since $X^2(t) \geq 0$, we must have $\mathsf{E}[X^2(t)] \geq 0$. For the second property, we substitute $u = t + \tau$ in Definition 13.13 to obtain

$$R_X(t,\tau) = \mathsf{E}\left[X(u-\tau)X(u)\right] = R_X(u,-\tau). \tag{13.46}$$

Since $X(t)$ is wide sense stationary,

$$R_X(t,\tau) = R_X(\tau) = R_X(u,-\tau) = R_X(-\tau). \tag{13.47}$$

The proof of the third property is a little more complex. First, we note that when $X(t)$ is wide sense stationary, $\mathsf{Var}[X(t)] = C_X(0)$, a constant for all t. Second, Theorem 5.14 implies that

$$C_X(t,\tau) \leq \sigma_{X(t)}\sigma_{X(t+\tau)} = C_X(0). \tag{13.48}$$

[Continued]

Proof: Theorem 13.12 (Continued 2)

Now, for any numbers a, b, and c, if $a \leq b$ and $c \geq 0$, then $(a+c)^2 \leq (b+c)^2$. Choosing $a = C_X(t,\tau)$, $b = C_X(0)$, and $c = \mu_X^2$ yields

$$\left(C_X(t,\tau) + \mu_X^2\right)^2 \leq \left(C_X(0) + \mu_X^2\right)^2. \tag{13.49}$$

In this expression, the left side equals $(R_X(\tau))^2$ and the right side is $(R_X(0))^2$, which proves the third part of the theorem. The proof for the random sequence X_n is essentially the same. Problem 13.9.10 asks the reader to confirm this fact.

Definition 13.16 Average Power

The average power *of a wide sense stationary process* $X(t)$ *is* $R_X(0) = E[X^2(t)]$.

. . . .

The average power *of a wide sense stationary sequence* X_n *is* $R_X[0] = E[X_n^2]$.

Theorem 13.13

Let $X(t)$ be a stationary random process with expected value μ_X and autocovariance $C_X(\tau)$. If $\int_{-\infty}^{\infty} |C_X(\tau)| \, d\tau < \infty$, then $\overline{X}(T), \overline{X}(2T), \ldots$ is an unbiased, consistent sequence of estimates of μ_X.

Proof: Theorem 13.13

First we verify that $\overline{X}(T)$ is unbiased:

$$\mathsf{E}\left[\overline{X}(T)\right] = \frac{1}{2T}\,\mathsf{E}\left[\int_{-T}^{T} X(t)\,dt\right] = \frac{1}{2T}\int_{-T}^{T} \mathsf{E}\left[X(t)\right]\,dt = \frac{1}{2T}\int_{-T}^{T} \mu_X\,dt = \mu_X. \qquad (13.52)$$

To show consistency, it is sufficient to show that $\lim_{T\to\infty} \mathsf{Var}[\overline{X}(T)] = 0$. First, we observe that $\overline{X}(T) - \mu_X = \frac{1}{2T}\int_{-T}^{T}(X(t) - \mu_X)\,dt$. This implies

$$\begin{aligned}
\mathsf{Var}[\overline{X}(T)] &= \mathsf{E}\left[\left(\frac{1}{2T}\int_{-T}^{T}(X(t)-\mu_X)\,dt\right)^2\right] \\
&= \mathsf{E}\left[\frac{1}{(2T)^2}\left(\int_{-T}^{T}(X(t)-\mu_X)\,dt\right)\left(\int_{-T}^{T}(X(t')-\mu_X)\,dt'\right)\right] \\
&= \frac{1}{(2T)^2}\int_{-T}^{T}\int_{-T}^{T}\mathsf{E}\left[(X(t)-\mu_X)(X(t')-\mu_X)\right]\,dt'\,dt \\
&= \frac{1}{(2T)^2}\int_{-T}^{T}\int_{-T}^{T} C_X(t'-t)\,dt'\,dt. \qquad (13.53)
\end{aligned}$$

[Continued]

Proof: Theorem 13.13 (Continued 2)

We note that

$$\int_{-T}^{T} C_X(t'-t)\,dt' \le \int_{-T}^{T} \left|C_X(t'-t)\right|\,dt'$$
$$\le \int_{-\infty}^{\infty} \left|C_X(t'-t)\right|\,dt' = \int_{-\infty}^{\infty} |C_X(\tau)|\,d\tau < \infty. \tag{13.54}$$

Hence there exists a constant K such that

$$\text{Var}[\overline{X}(T)] \le \frac{1}{(2T)^2}\int_{-T}^{T} K\,dt = \frac{K}{2T}. \tag{13.55}$$

Thus $\lim_{T\to\infty} \text{Var}[\overline{X}(T)] \le \lim_{T\to\infty} \frac{K}{2T} = 0$.

Quiz 13.9

Which of the following functions are valid autocorrelation functions?

(a) $R_1(\tau) = e^{-|\tau|}$

(b) $R_2(\tau) = e^{-\tau^2}$

(c) $R_3(\tau) = e^{-\tau} \cos \tau$

(d) $R_4(\tau) = e^{-\tau^2} \sin \tau$

Quiz 13.9 Solution

We must check whether each function $R(\tau)$ meets the conditions of Theorem 13.12:

$$R(\tau) \geq 0, \qquad R(\tau) = R(-\tau), \qquad |R(\tau)| \leq R(0). \tag{1}$$

(a) $R_1(\tau) = e^{-|\tau|}$ meets all three conditions and thus is valid.

(b) $R_2(\tau) = e^{-\tau^2}$ also is valid.

(c) $R_3(\tau) = e^{-\tau} \cos \tau$ is not valid because

$$R_3(-2\pi) = e^{2\pi} \cos 2\pi = e^{2\pi} > 1 = R_3(0) \tag{2}$$

(d) $R_4(\tau) = e^{-\tau^2} \sin \tau$ also cannot be an autocorrelation function because

$$R_4(\pi/2) = e^{-\pi/2} \sin \pi/2 = e^{-\pi/2} > 0 = R_4(0) \tag{3}$$

Cross-Correlation

Definition 13.17 Cross-Correlation Function

The cross-correlation *of continuous-time random processes* $X(t)$ *and* $Y(t)$ *is*

$$R_{XY}(t, \tau) = \mathsf{E}\left[X(t)Y(t + \tau)\right].$$

. . . .

The cross-correlation *of random sequences* X_n *and* Y_n *is*

$$R_{XY}[m, k] = \mathsf{E}\left[X_m Y_{m+k}\right].$$

Definition 13.18 Jointly Wide Sense Stationary Processes

Continuous-time random processes $X(t)$ *and* $Y(t)$ *are* jointly wide sense stationary *if* $X(t)$ *and* $Y(t)$ *are both wide sense stationary, and the cross-correlation depends only on the time difference between the two random variables:*

$$R_{XY}(t, \tau) = R_{XY}(\tau).$$

. . . .

Random sequences X_n *and* Y_n *are* jointly wide sense stationary *if* X_n *and* Y_n *are both wide sense stationary and the cross-correlation depends only on the index difference between the two random variables:*

$$R_{XY}[m, k] = R_{XY}[k].$$

Example 13.18 Problem

Suppose we are interested in $X(t)$ but we can observe only

$$Y(t) = X(t) + N(t), \tag{13.56}$$

where $N(t)$ is a noise process that interferes with our observation of $X(t)$. Assume $X(t)$ and $N(t)$ are independent wide sense stationary processes with $\mathrm{E}[X(t)] = \mu_X$ and $\mathrm{E}[N(t)] = \mu_N = 0$. Is $Y(t)$ wide sense stationary? Are $X(t)$ and $Y(t)$ jointly wide sense stationary? Are $Y(t)$ and $N(t)$ jointly wide sense stationary?

Example 13.18 Solution

Since the expected value of a sum equals the sum of the expected values,

$$\mathsf{E}[Y(t)] = \mathsf{E}[X(t)] + \mathsf{E}[N(t)] = \mu_X. \tag{13.57}$$

Next, we must find the autocorrelation

$$\begin{aligned} R_Y(t,\tau) &= \mathsf{E}[Y(t)Y(t+\tau)] \\ &= \mathsf{E}[(X(t)+N(t))(X(t+\tau)+N(t+\tau))] \\ &= R_X(\tau) + R_{XN}(t,\tau) + R_{NX}(t,\tau) + R_N(\tau). \end{aligned} \tag{13.58}$$

Since $X(t)$ and $N(t)$ are independent, $R_{NX}(t,\tau) = \mathsf{E}[N(t)]\,\mathsf{E}[X(t+\tau)] = 0$. Similarly, $R_{XN}(t,\tau) = \mu_X \mu_N = 0$. This implies

$$R_Y(t,\tau) = R_X(\tau) + R_N(\tau). \tag{13.59}$$

The right side of this equation indicates that $R_Y(t,\tau)$ depends only on τ, which implies that $Y(t)$ is wide sense stationary. To determine whether $Y(t)$ and $X(t)$ are jointly wide sense stationary, we calculate the cross-correlation

$$\begin{aligned} R_{YX}(t,\tau) = \mathsf{E}[Y(t)X(t+\tau)] &= \mathsf{E}[(X(t)+N(t))X(t+\tau)] \\ &= R_X(\tau) + R_{NX}(t,\tau) = R_X(\tau). \end{aligned} \tag{13.60}$$

We can conclude that $X(t)$ and $Y(t)$ are jointly wide sense stationary. Similarly, we can verify that $Y(t)$ and $N(t)$ are jointly wide sense stationary by calculating

$$\begin{aligned} R_{YN}(t,\tau) = \mathsf{E}[Y(t)N(t+\tau)] &= \mathsf{E}[(X(t)+N(t))N(t+\tau)] \\ &= R_{XN}(t,\tau) + R_N(\tau) = R_N(\tau). \end{aligned} \tag{13.61}$$

Example 13.19 Problem

X_n is a wide sense stationary random sequence with autocorrelation function $R_X[k]$. The random sequence Y_n is obtained from X_n by reversing the sign of every other random variable in X_n: $Y_n = -1^n X_n$.

(a) Express the autocorrelation function of Y_n in terms of $R_X[k]$.

(b) Express the cross-correlation function of X_n and Y_n in terms of $R_X[k]$.

(c) Is Y_n wide sense stationary?

(d) Are X_n and Y_n jointly wide sense stationary?

Example 13.19 Solution

The autocorrelation function of Y_n is

$$R_Y[n,k] = \mathsf{E}\left[Y_n Y_{n+k}\right] = \mathsf{E}\left[(-1)^n X_n (-1)^{n+k} X_{n+k}\right]$$
$$= (-1)^{2n+k}\,\mathsf{E}\left[X_n X_{n+k}\right]$$
$$= (-1)^k R_X[k]. \qquad (13.62)$$

Y_n is wide sense stationary because the autocorrelation depends only on the index difference k. The cross-correlation of X_n and Y_n is

$$R_{XY}[n,k] = \mathsf{E}\left[X_n Y_{n+k}\right] = \mathsf{E}\left[X_n (-1)^{n+k} X_{n+k}\right]$$
$$= (-1)^{n+k}\,\mathsf{E}\left[X_n X_{n+k}\right]$$
$$= (-1)^{n+k} R_X[k]. \qquad (13.63)$$

X_n and Y_n are not jointly wide sense stationary because the cross-correlation depends on both n and k. When n and k are both even or when n and k are both odd, $R_{XY}[n,k] = R_X[k]$; otherwise $R_{XY}[n,k] = -R_X[k]$.

Theorem 13.14

If $X(t)$ and $Y(t)$ are jointly wide sense stationary continuous-time processes, then

$$R_{XY}(\tau) = R_{YX}(-\tau).$$

....

If X_n and Y_n are jointly wide sense stationary random sequences, then

$$R_{XY}[k] = R_{YX}[-k].$$

Proof: Theorem 13.14

From Definition 13.17, $R_{XY}(\tau) = \mathsf{E}[X(t)Y(t+\tau)]$. Making the substitution $u = t + \tau$ yields

$$R_{XY}(\tau) = \mathsf{E}\left[X(u-\tau)Y(u)\right] = \mathsf{E}\left[Y(u)X(u-\tau)\right] = R_{YX}(u, -\tau). \quad (13.64)$$

Since $X(t)$ and $Y(t)$ are jointly wide sense stationary,

$$R_{YX}(u, -\tau) = R_{YX}(-\tau).$$

The proof is similar for random sequences.

Quiz 13.10

$X(t)$ is a wide sense stationary stochastic process with autocorrelation function $R_X(\tau)$. $Y(t)$ is identical to $X(t)$, except that time is reversed: $Y(t) = X(-t)$.

(a) Express the autocorrelation function of $Y(t)$ in terms of $R_X(\tau)$. Is $Y(t)$ wide sense stationary?

(b) Express the cross-correlation function of $X(t)$ and $Y(t)$ in terms of $R_X(\tau)$. Are $X(t)$ and $Y(t)$ jointly wide sense stationary?

Quiz 13.10 Solution

(a) The autocorrelation of $Y(t)$ is

$$\begin{aligned} R_Y(t,\tau) &= \mathsf{E}\left[Y(t)Y(t+\tau)\right] \\ &= \mathsf{E}\left[X(-t)X(-t-\tau)\right] \\ &= R_X(-t-(-t-\tau)) = R_X(\tau). \end{aligned} \tag{1}$$

Since $\mathsf{E}[Y(t)] = \mathsf{E}[X(-t)] = \mu_X$, we can conclude that $Y(t)$ is a wide sense stationary process. In fact, we see that by viewing a process backwards in time, we see the same second order statistics.

(b) Since $X(t)$ and $Y(t)$ are both wide sense stationary processes, we can check whether they are jointly wide sense stationary by seeing if $R_{XY}(t,\tau)$ is just a function of τ. In this case,

$$\begin{aligned} R_{XY}(t,\tau) &= \mathsf{E}\left[X(t)Y(t+\tau)\right] \\ &= \mathsf{E}\left[X(t)X(-t-\tau)\right] \\ &= R_X(t-(-t-\tau)) = R_X(2t+\tau). \end{aligned} \tag{2}$$

Since $R_{XY}(t,\tau)$ depends on both t and τ, we conclude that $X(t)$ and $Y(t)$ are not jointly wide sense stationary. To see why this is, suppose $R_X(\tau) = e^{-|\tau|}$ so that samples of $X(t)$ far apart in time have almost no correlation. In this case, as t gets larger, $Y(t) = X(-t)$ and $X(t)$ become less correlated.

Section 13.11

Gaussian Processes

Definition 13.19 Gaussian Process

$X(t)$ *is a Gaussian stochastic process if and only if* $\mathbf{X} = \begin{bmatrix} X(t_1) & \cdots & X(t_k) \end{bmatrix}'$ *is a Gaussian random vector for any integer* $k > 0$ *and any set of time instants* $t_1, t_2, \ldots, t_k$.

. . . .

X_n *is a Gaussian random sequence if and only if* $\mathbf{X} = \begin{bmatrix} X_{n_1} & \cdots & X_{n_k} \end{bmatrix}'$ *is a Gaussian random vector for any integer* $k > 0$ *and any set of time instants* $n_1, n_2, \ldots, n_k$.

Theorem 13.15

If $X(t)$ is a wide sense stationary Gaussian process, then $X(t)$ is a stationary Gaussian process.

. . . .

If X_n is a wide sense stationary Gaussian sequence, X_n is a stationary Gaussian sequence.

Proof: Theorem 13.15

Let $\boldsymbol{\mu}$ and $\mathbf{C}$ denote the expected value vector and the covariance matrix of the random vector $\mathbf{X} = \begin{bmatrix} X(t_1) & \ldots & X(t_k) \end{bmatrix}'$. Let $\bar{\boldsymbol{\mu}}$ and $\bar{\mathbf{C}}$ denote the same quantities for the time-shifted random vector

$$\bar{\mathbf{X}} = \begin{bmatrix} X(t_1 + T) & \cdots & X(t_k + T) \end{bmatrix}'.$$

Since $X(t)$ is wide sense stationary, $\mathrm{E}[X(t_i)] = \mathrm{E}[X(t_i + T)] = \mu_X$. The i, jth entry of $\mathbf{C}$ is

$$\begin{aligned} C_{ij} = C_X(t_i, t_j) &= C_X(t_j - t_i) \\ &= C_X(t_j + T - (t_i + T)) = C_X(t_i + T, t_j + T) = \bar{C}_{ij}. \end{aligned} \tag{13.65}$$

Thus $\boldsymbol{\mu} = \bar{\boldsymbol{\mu}}$ and $\mathbf{C} = \bar{\mathbf{C}}$, implying that $f_{\mathbf{X}}(\mathbf{x}) = f_{\bar{\mathbf{X}}}(\mathbf{x})$. Hence $X(t)$ is a stationary process. The same reasoning applies to a Gaussian random sequence X_n.

Definition 13.20 White Gaussian Noise

$W(t)$ is a white Gaussian noise process if and only if $W(t)$ is a stationary Gaussian stochastic process with the properties $\mu_W = 0$ and $R_W(\tau) = \eta_0 \delta(\tau)$.

Quiz 13.11

$X(t)$ is a stationary Gaussian random process with $\mu_X(t) = 0$ and auto-correlation function $R_X(\tau) = 2^{-|\tau|}$. What is the joint PDF of $X(t)$ and $X(t+1)$?

Quiz 13.11 Solution

From the problem statement,

$$E[X(t)] = E[X(t+1)] = 0, \tag{1}$$

$$E[X(t)X(t+1)] = 1/2, \tag{2}$$

$$\text{Var}[X(t)] = \text{Var}[X(t+1)] = 1. \tag{3}$$

The Gaussian random vector $\mathbf{X} = \begin{bmatrix} X(t) & X(t+1) \end{bmatrix}'$ has covariance matrix and corresponding inverse

$$\mathbf{C_X} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}, \qquad \mathbf{C_X^{-1}} = \frac{4}{3}\begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}. \tag{4}$$

Since

$$\mathbf{x}'\mathbf{C_X^{-1}x} = \begin{bmatrix} x_0 & x_1 \end{bmatrix}' \frac{4}{3}\begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \frac{4}{3}\left(x_0^2 - x_0 x_+ x_1^2\right), \tag{5}$$

the joint PDF of $X(t)$ and $X(t+1)$ is the Gaussian vector PDF

$$f_{X(t),X(t+1)}(x_0, x_1) = \frac{1}{(2\pi)^{n/2}[\det(\mathbf{C_X})]^{1/2}} \exp\left(-\frac{1}{2}\mathbf{x}'\mathbf{C_X^{-1}x}\right) \tag{6}$$

$$= \frac{1}{\sqrt{3\pi^2}} e^{-\frac{2}{3}(x_0^2 - x_0 x_1 + x_1^2)}. \tag{7}$$

Section 13.12

Matlab

Example 13.20 Problem

Use Matlab to generate the arrival times $S_1, S_2, \ldots$ of a rate λ Poisson process over a time interval $[0, T]$.

Example 13.20 Solution

```
function s=poissonarrivals(lam,T)
%arrival times s=[s(1) ... s(n)]
%  s(n)<= T < s(n+1)
n=ceil(1.1*lam*T);
s=cumsum(exponentialrv(lam,n));
while (s(length(s))< T),
  s_new=s(length(s))+ ...
    cumsum(exponentialrv(lam,n));
  s=[s; s_new];
end
s=s(s<=T);
```

To generate Poisson arrivals at rate λ, we employ Theorem 13.4, which says that the interarrival times are independent exponential (λ) random variables. Given interarrival times X_i, the ith arrival time is the cumulative sum

$$S_i = X_1 + X_2 + \cdots + X_i.$$

The function `poissonarrivals` outputs the cumulative sums of independent exponential random variables; it returns the vector `s` with `s(i)` corresponding to S_i, the ith arrival time. Note that the length of `s` is a Poisson (λT) random variable because the number of arrivals in $[0, T]$ is random.

Example 13.21 Problem

Generate a sample path of $N(t)$, a rate $\lambda = 5$ arrivals/min Poisson process. Plot $N(t)$ over a 10-minute interval.

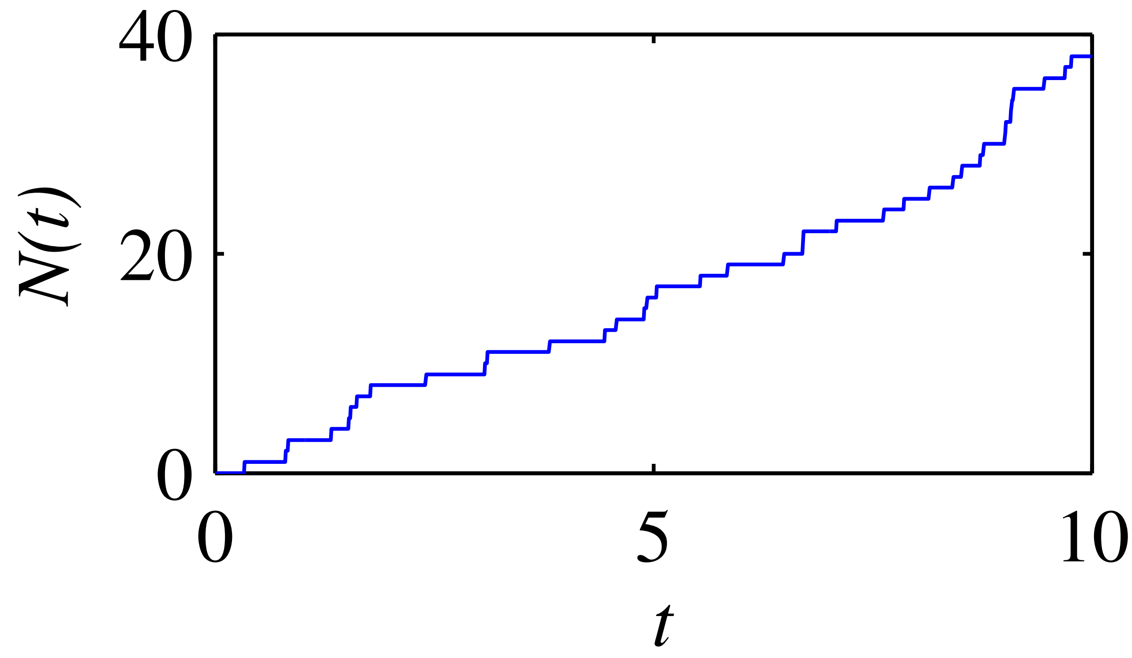
Example 13.21 Solution

```
function N=poissonprocess(lambda,t)
%N(i) = no. of arrivals by t(i)
s=poissonarrivals(lambda,max(t));
N=count(s,t);
```

Given `t` $= \begin{bmatrix} t_1 & \cdots & t_m \end{bmatrix}'$, the function `poissonprocess` generates the vector $\mathbf{N} = \begin{bmatrix} N_1 & \cdots & N_m \end{bmatrix}'$ where $N_i = N(t_i)$ for a rate λ Poisson process $N(t)$. The basic idea of `poissonprocess.m` is that given the arrival times $S_1, S_2, \ldots$, $N(t) = \max\{n | S_n \leq t\}$ is the number of arrivals that occur by time t. In particular, in `N=count(s,t)`, `N(i)` is the number of elements of `s` that are less than or equal to `t(i)`. A sample path generated by `poissonprocess.m` appears in Figure 13.5.

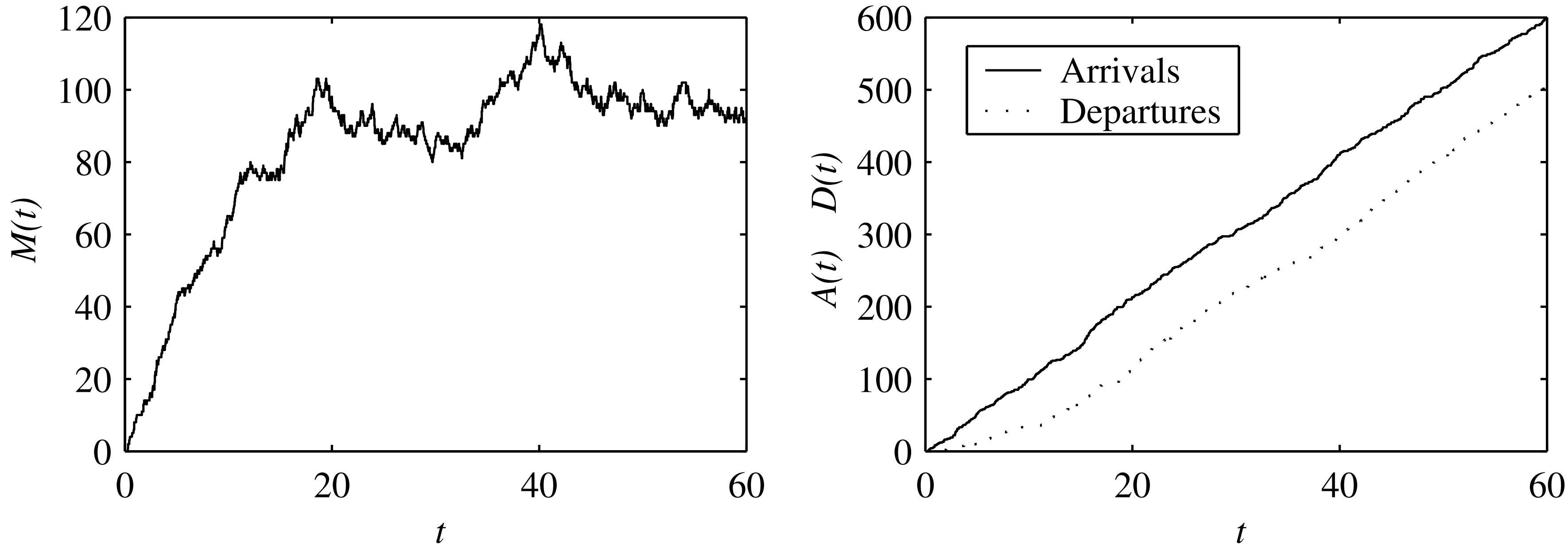
Figure 13.5

```
t=0.01*(0:1000);
lambda=5;
N=poissonprocess(lambda,t);
plot(t,N)
xlabel('\it t');
ylabel('\it N(t)');
```



A Poisson process sample path $N(t)$ generated by `poissonprocess.m`.

Figure 13.6



(a) **(b)**

Output for `simswitch`: **(a)** The active-call process $M(t)$. **(b)** The arrival and departure processes $A(t)$ and $D(t)$ such that $M(t) = A(t) - D(t)$.

Example 13.22 Problem

Simulate 60 minutes of activity of the telephone switch of Example 13.4 under the following assumptions.

(a) The switch starts with $M(0) = 0$ calls.

(b) Arrivals occur as a Poisson process of rate $\lambda = 10$ calls/min.

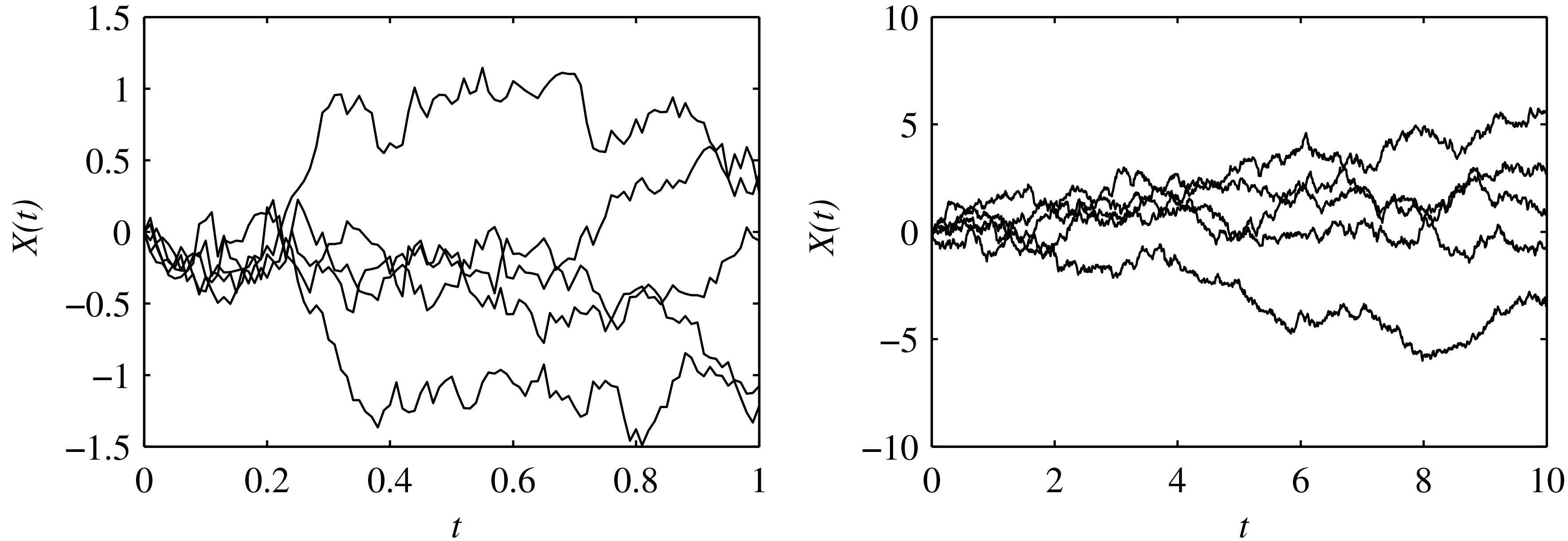
(c) The duration of each call (often called the holding time) in minutes is an exponential (1/10) random variable independent of the number of calls in the system and the duration of any other call.

Example 13.22 Solution

```
function M=simswitch(lambda,mu,t)
%Poisson arrivals, rate lambda
%Exponential (mu) call duration
%For vector t of times
%M(i) = no. of calls at time t(i)
s=poissonarrivals(lambda,max(t));
y=s+exponentialrv(mu,length(s));
A=count(s,t);
D=count(y,t);
M=A-D;
```

In `simswitch.m`, the vectors `s` and `x` mark the arrival times and call durations. The ith call arrives at time `s(i)`, stays for time `x(i)`, and departs at time `y(i)=s(i)`. Thus the vector `y=s+x` denotes the call completion times, also known as *departures*. By counting the arrivals `s` and departures `y`, we produce the arrival and departure processes `A` and `D`. At any given time t, the number of calls in the system equals the number of arrivals minus the number of departures. Hence `M=A-D` is the number of calls in the system. One run of `simswitch.m` depicting sample functions of $A(t)$, $D(t)$, and $M(t) = A(t) - D(t)$ appears in Figure 13.6.

Figure 13.7



Sample paths of Brownian motion.

Example 13.23 Problem

Generate a Brownian motion process $W(t)$ with parameter α.

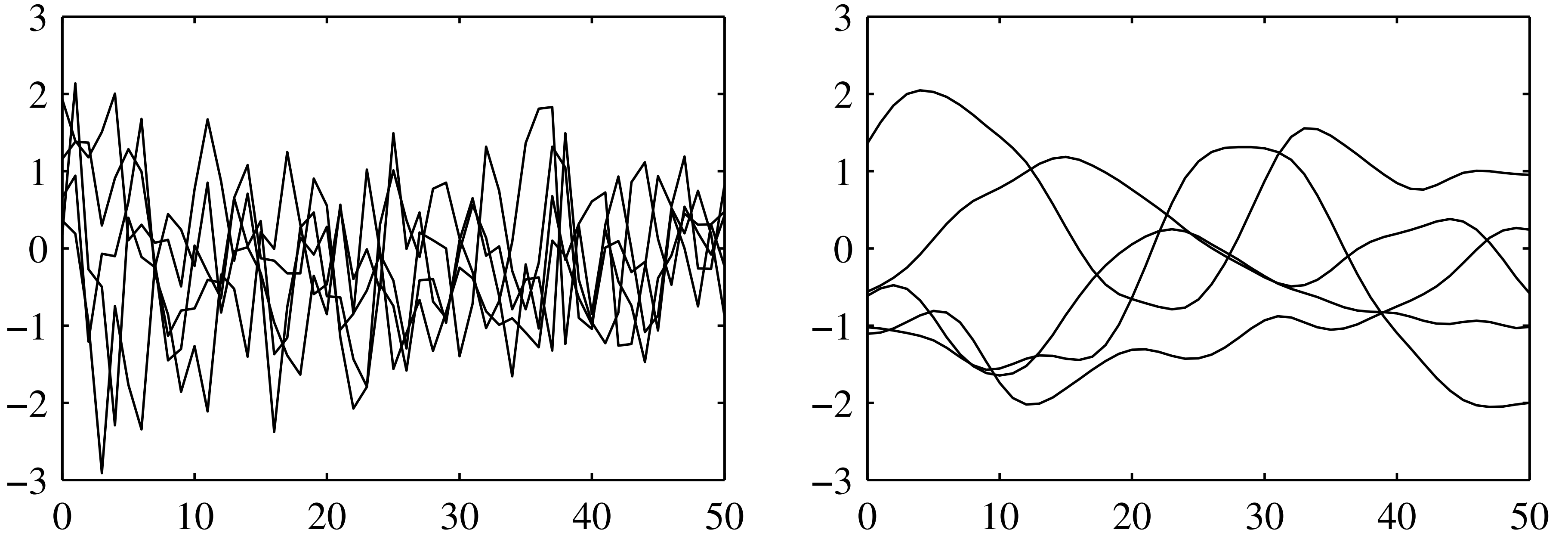
Example 13.23 Solution

```
function w=brownian(alpha,t)
%Brownian motion process
%sampled at t(1)<t(2)< ...
t=t(:);
n=length(t);
delta=t-[0;t(1:n-1)];
x=sqrt(alpha*delta).*gaussrv(0,1,n);
w=cumsum(x);
```

The function `brownian` outputs a Brownian motion process $W(t)$ sampled at times $t_i =$ `t(i)`. The vector `x` consists of the independent increments `x(i)` scaled to have variance $\alpha(t_i - t_{i-1})$.

Each graph in Figure 13.7 shows five sample paths of a Brownian motion processes with $\alpha = 1$. For plot (a), $0 \le t \le 1$, for plot (b), $0 \le t \le 10$. Note that the plots have different y-axis scaling because $\text{Var}[X(t)] = \alpha t$. Thus as time increases, the excursions from the origin tend to get larger.

Figure 13.8



(a) $a = 1$: `gseq(1,50,5)` **(b)** $a = 0.01$: `gseq(0.01,50,5)`

Two sample outputs for Example 13.24.

Example 13.24 Problem

Write a Matlab function `x=gseq(a,n,m)` that generates m sample vectors $\mathbf{X} = \begin{bmatrix} X_0 & \cdots & X_n \end{bmatrix}'$ of a stationary Gaussian sequence with

$$\mu_X = 0, \qquad C_X[k] = \frac{1}{1+ak^2}. \tag{13.68}$$

Example 13.24 Solution

```
function x=gseq(a,n,m)
nn=0:n;
cx=1./(1+a*nn.^2);
x=gaussvector(0,cx,m);
plot(nn,x);
```

All we need to do is generate the vector `cx` corresponding to the covariance function. Figure 13.8 shows sample outputs for graphs
(a) $a = 1$: `gseq(1,50,5)`,
(b) $a = 0.01$: `gseq(0.01,50,5)`.
We observe in Figure 13.8 that each graph shows $m = 5$ sample paths even though graph (a) may appear to have many more. The graphs look very different because for $a = 1$, samples just a few steps apart are nearly uncorrelated and the sequence varies quickly with time. That is, the sample paths in graph (a) zig-zag around. By contrast, when $a = 0.01$, samples have significant correlation and the sequence varies slowly. that is, in graph (b), the sample paths look realtively smooth.

Quiz 13.12

The switch simulation of Example 13.22 is unrealistic in the assumption that the switch can handle an arbitrarily large number of calls. Modify the simulation so that the switch blocks (i.e., discards) new calls when the switch has $c = 120$ calls in progress. Estimate $\mathrm{P}[B]$, the probability that a new call is blocked. Your simulation may need to be significantly longer than 60 minutes.

Quiz 13.12 Solution

The simple structure of the switch simulation of Example 13.22 admits a deceptively simple solution in terms of the vector of arrivals A and the vector of departures D. With the introduction of call blocking. we cannot generate these vectors all at once. In particular, when an arrival occurs at time t, we need to know that $M(t)$, the number of ongoing calls, satisfies $M(t) < c = 120$. Otherwise, when $M(t) = c$, we must block the call. Call blocking can be implemented by setting the service time of the call to zero so that the call departs as soon as it arrives.

The blocking switch is an example of a discrete event system. The system evolves via a sequence of discrete events, namely arrivals and departures, at discrete time instances. A simulation of the system moves from one time instant to the next by maintaining a chronological schedule of future events (arrivals and departures) to be executed. The program simply executes the event at the head of the schedule. The logic of such a simulation is [Continued]

Quiz 13.12 Solution (Continued 2)

1. Start at time $t = 0$ with an empty system. Schedule the first arrival to occur at S_1, an exponential (λ) random variable.

2. Examine the head-of-schedule event.

   - When the head-of-schedule event is the kth arrival is at time t, check the state $M(t)$.
     - If $M(t) < c$, admit the arrival, increase the system state n by 1, and schedule a departure to occur at time $t + S_n$, where S_k is an exponential (λ) random variable.
     - If $M(t) = c$, block the arrival, do not schedule a departure event.
   - If the head of schedule event is a departure, reduce the system state n by 1.

3. Delete the head-of-schedule event and go to step 2.

After the head-of-schedule event is completed and any new events (departures in this system) are scheduled, we know the system state cannot change until the next scheduled event. Thus we know that $M(t)$ will stay the same until then. In our simulation, we use the vector `t` as the set of time instances at which we inspect the system state. Thus for all times `t(i)` between the current head-of-schedule event and the next, we set `m(i)` to the current switch state. [Continued]

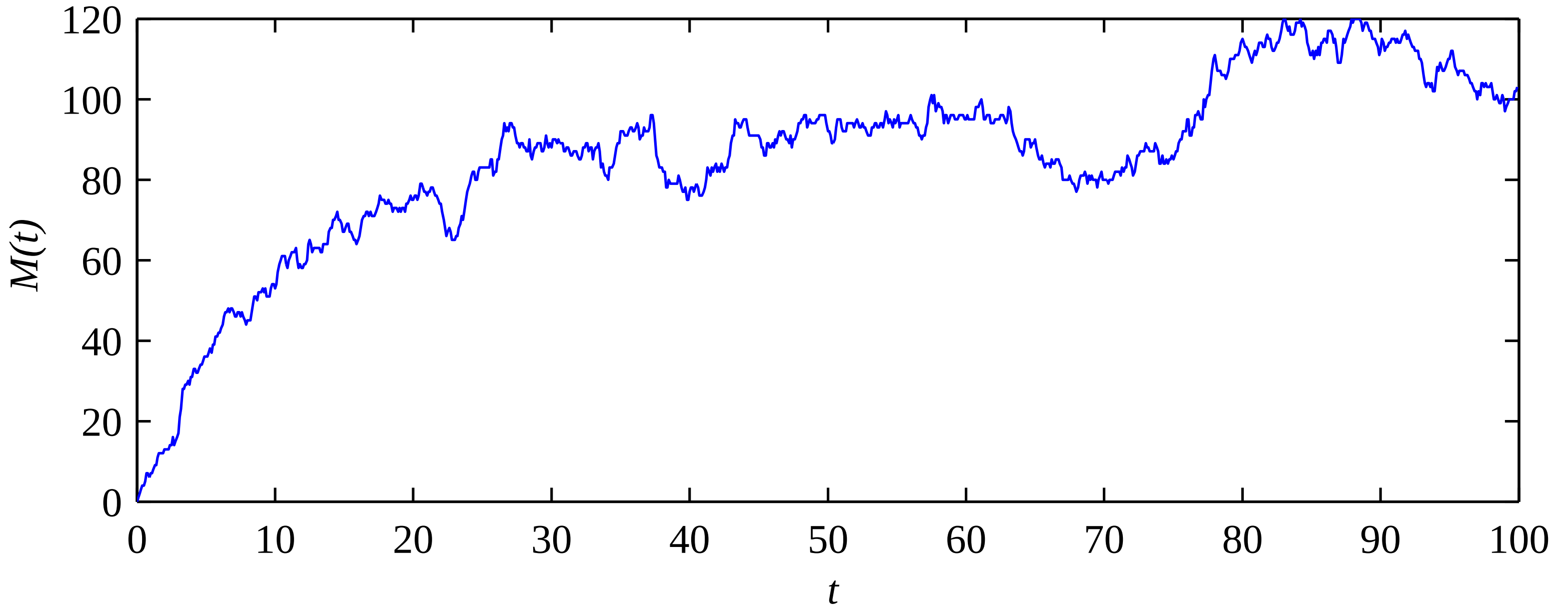
The program `simblockswitch.m` can be found in the regular solution manual.In most programming languages, it is common to implement the event schedule as a linked list where each item in the list has a data structure indicating an event timestamp and the type of the event. In Matlab, a simple (but not elegant) way to do this is to have maintain two vectors: `time` is a list of timestamps of scheduled events and `event` is a the list of event types. In this case, `event(i)=1` if the ith scheduled event is an arrival, or `event(i)=-1` if the ith scheduled event is a departure.

When the program is passed a vector `t`, the output `[m a b]` is such that `m(i)` is the number of ongoing calls at time `t(i)` while `a` and `b` are the number of admits and blocks. The following instructions

```
t=0:0.1:5000;
[m,a,b]=simblockswitch(10,0.1,120,t);
plot(t,m);
```

generated a simulation lasting 5,000 minutes. Here is a sample path of the first 100 minutes of that simulation: [Continued]

Quiz 13.12 Solution (Continued 4)



The 5,000 minute full simulation produced `a=49658` admitted calls and `b=239` blocked calls. We can estimate the probability a call is blocked as

$$\hat{P}_b = \frac{b}{a+b} = 0.0048. \tag{1}$$

In the Markov Chains Supplement, we will learn that the exact blocking probability is given by the "Erlang-B formula." From the Erlang-B formula, we can calculate that the exact blocking probability is $P_b = 0.0057$. [Continued]

Quiz 13.12 Solution (Continued 5)

One reason our simulation underestimates the blocking probability is that in a 5,000 minute simulation, roughly the first 100 minutes are needed to load up the switch since the switch is idle when the simulation starts at time $t = 0$. However, this says that roughly the first two percent of the simulation time was unusual. Thus this would account for only part of the disparity. The rest of the gap between 0.0048 and 0.0057 is that a simulation that includes only 239 blocks is not all that likely to give a very accurate result for the blocking probability.