PMF of a Function of Two Discrete Random Variables

Theorem 6.1

For discrete random variables X and Y, the derived random variable $W = g(X, Y)$ has PMF

$$P_W(w) = \sum_{(x,y):g(x,y)=w} P_{X,Y}(x, y) \, .$$

Example 6.1 Problem

$P_{L,X}(l,x)$	$x = 40$	$x = 60$
$l = 1$	0.15	0.1
$l = 2$	0.3	0.2
$l = 3$	0.15	0.1

A firm sends out two kinds of newsletters. One kind contains only text and grayscale images and requires 40 cents to print each page. The other kind contains color pictures that cost 60 cents per page. Newsletters can be 1, 2, or 3 pages long. Let the random variable L represent the length of a newsletter in pages. $S_L = \{1, 2, 3\}$. Let the random variable X represent the cost in cents to print each page. $S_X = \{40, 60\}$. After observing many newsletters, the firm has derived the probability model shown above. Let $W = g(L, X) = LX$ be the total cost in cents of a newsletter. Find the range S_W and the PMF $P_W(w)$.

Example 6.1 Solution

$P_{L,X}(l,x)$	$x = 40$	$x = 60$
$l = 1$	0.15 $(W{=}40)$	0.1 $(W{=}60)$
$l = 2$	0.3 $(W{=}80)$	0.2 $(W{=}120)$
$l = 3$	0.15 $(W{=}120)$	0.1 $(W{=}180)$

For each of the six possible combinations of L and X, we record $W = LX$ under the corresponding entry in the PMF table on the left. The range of W is

$$S_W = \{40, 60, 80, 120, 180\}$$

. With the exception of $W = 120$, there is a unique pair L, X such that $W = LX$. For $W = 120$,

· ·

w	40	60	80	120	180
$P_W(w)$	0.15	0.1	0.3	0.35	0.1

$$P_W(120) = P_{L,X}(3, 40) + P_{L,X}(2, 60)\,.$$

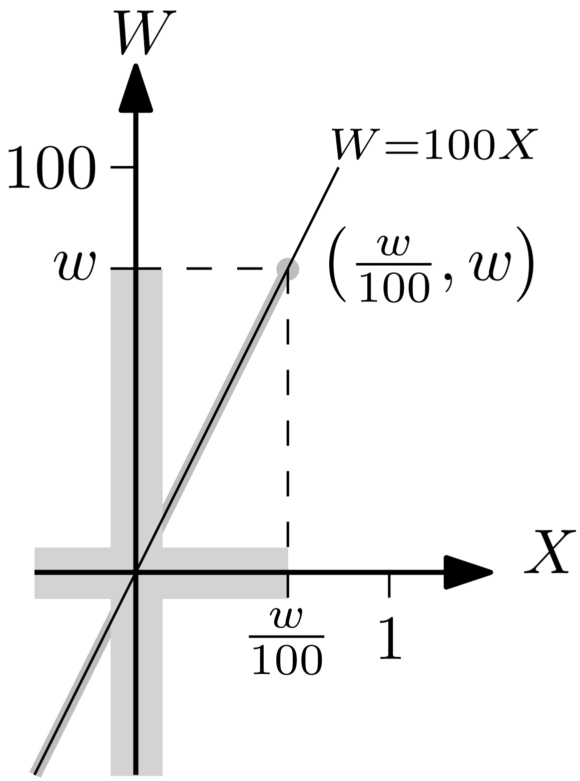
The corresponding probabilities are recorded in the second table on the left.

Functions Yielding Continuous Random Variables

Example 6.2 Problem

In Example 4.2, W centimeters is the location of the pointer on the 1-meter circumference of the circle. Use the solution of Example 4.2 to derive $f_W(w)$.

Example 6.2 Solution



The function $W = 100X$, where X in Example 4.2 is the location of the pointer measured in meters. To find the CDF $F_W(w) = \mathrm{P}[W \leq w]$, the first step is to translate the event $\{W \leq w\}$ into an event described by X. Each outcome of the experiment is mapped to an (X, W) pair on the line $W = 100X$. Thus the event $\{W \leq w\}$, shown with gray highlight on the vertical axis, is the same event as $\{X \leq w/100\}$, which is shown with gray highlight on the horizontal axis. Both of these events correspond in the figure to observing an (X, W) pair along the highlighted section of the line $w = g(X) = 100w$. This translation of the event $W = w$ to an event described in terms of X depends only on the function $g(X)$. Specifically, it does not depend on the probability model for X. From the figure, we see that

$$F_W(w) = \mathrm{P}[W \leq w] = \mathrm{P}[100X \leq w] = \mathrm{P}[X \leq w/100] = F_X(w/100). \tag{6.1}$$

[Continued]

Example 6.2 Solution (Continued 2)

The calculation of $F_X(w/100)$ depends on the probability model for X. For this problem, we recall that Example 4.2 derives the CDF of X,

$$F_X(x) = \begin{cases} 0 & x < 0, \\ x & 0 \le x < 1, \\ 1 & x \ge 1. \end{cases} \tag{6.2}$$

From this result, we can use algebra to find

$$F_W(w) = F_X\left(\frac{w}{100}\right) = \begin{cases} 0 & \frac{w}{100} < 0, \\ \frac{w}{100} & 0 \le \frac{w}{100} < 1, \\ 1 & \frac{w}{100} \ge 1, \end{cases} = \begin{cases} 0 & w < 0, \\ \frac{w}{100} & 0 \le w < 100, \\ 1 & w \ge 100. \end{cases} \tag{6.3}$$

We take the derivative of the CDF of W over each of the intervals to find the PDF:

$$f_W(w) = \frac{dF_W(w)}{dw} = \begin{cases} 1/100 & 0 \le w < 100, \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

We see that W is the uniform $(0, 100)$ random variable.

Theorem 6.2

If $W = aX$, where $a > 0$, then W has CDF and PDF

$$F_W(w) = F_X(w/a)\,, \qquad f_W(w) = \frac{1}{a} f_X(w/a)\,.$$

Proof: Theorem 6.2

First, we find the CDF of W,

$$F_W(w) = \mathrm{P}[aX \le w] = \mathrm{P}[X \le w/a] = F_X(w/a)\,. \qquad (6.5)$$

We take the derivative of $F_Y(y)$ to find the PDF:

$$f_W(w) = \frac{dF_W(w)}{dw} = \frac{1}{a} f_X(w/a)\,. \qquad (6.6)$$

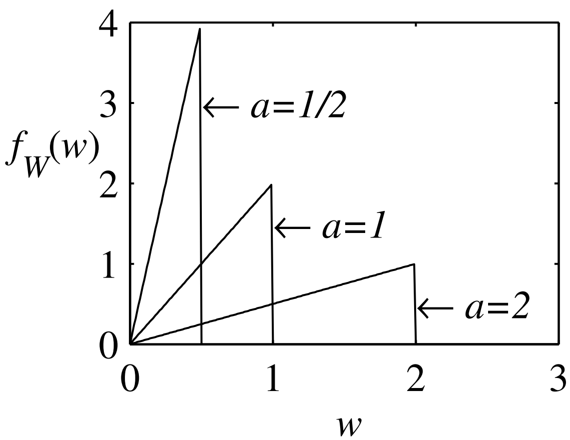
Example 6.3 Problem

The triangular PDF of X is

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.7}$$

Find the PDF of $W = aX$. Sketch the PDF of W for $a = 1/2, 1, 2$.

Example 6.3 Solution



For any $a > 0$, we use Theorem 6.2 to find the PDF:

$$f_W(w) = \frac{1}{a} f_X(w/a)$$

$$= \begin{cases} 2w/a^2 & 0 \leq w \leq a, \\ 0 & \text{otherwise.} \end{cases} \tag{6.8}$$

As a increases, the PDF stretches horizontally.

Theorem 6.3

$W = aX$, where $a > 0$.

(a) If X is uniform (b, c), then W is uniform (ab, ac).

(b) If X is exponential (λ), then W is exponential (λ/a).

(c) If X is Erlang (n, λ), then W is Erlang $(n, \lambda/a)$.

(d) If X is Gaussian (μ, σ), then W is Gaussian $(a\mu, a\sigma)$.

Theorem 6.4

If $W = X + b$,

$$F_W(w) = F_X(w - b), \qquad f_W(w) = f_X(w - b).$$

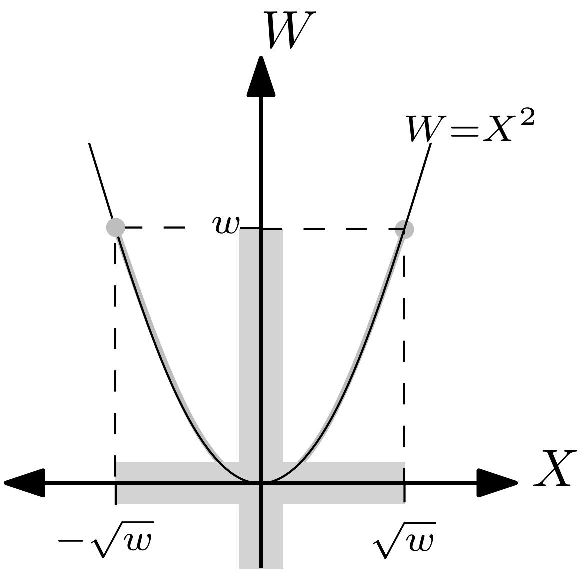
Proof: Theorem 6.4

First, we find the CDF $F_W(w) = \mathrm{P}[X + b \leq w] = \mathrm{P}[X \leq w - b] = F_X(w - b)$. We take the derivative of $F_W(w)$ to find the PDF: $f_W(w) = dF_W(w)/dw = f_X(w - b)$.

Example 6.4 Problem

Suppose X is the continuous uniform $(-1, 1)$ random variable and $W = X^2$. Find the CDF $F_W(w)$ and PDF $f_W(w)$.

Example 6.4 Solution



Although X can be negative, W is always nonnegative. Thus $F_W(w) = 0$ for $w < 0$. To find the CDF $F_W(w)$ for $w \geq 0$, the figure on the left shows that the event $\{W \leq w\}$, marked with gray highlight on the vertical axis, is the same as the event $\{-\sqrt{w} \leq X \leq \sqrt{w}\}$ marked on the horizontal axis. Both events correspond to (X, W) pairs on the highlighted segment of the function $W = g(X)$. The corresponding algebra is

$$F_W(w) = \mathrm{P}\left[X^2 \leq w\right] = \mathrm{P}\left[-\sqrt{w} \leq X \leq \sqrt{w}\right]. \tag{6.9}$$

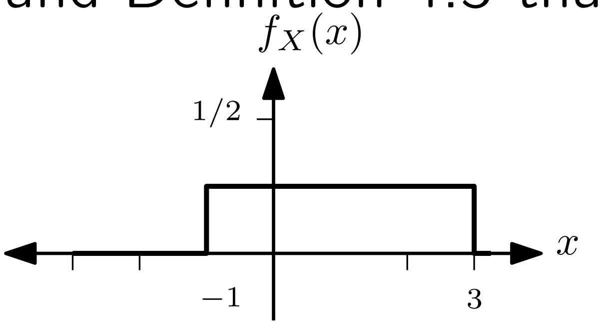
[Continued]

Example 6.4 Solution (Continued 2)

We can take one more step by writing the probability (6.9) as an integral using the PDF $f_X(x)$:

$$F_W(w) = \mathsf{P}\left[-\sqrt{w} \le X \le \sqrt{w}\right] = \int_{-\sqrt{w}}^{\sqrt{w}} f_X(x)\, dx. \tag{6.10}$$

So far, we have used no properties of the PDF $f_X(x)$. However, to evaluate the integral (6.10), we now recall from the problem statement and Definition 4.5 that the PDF of X is

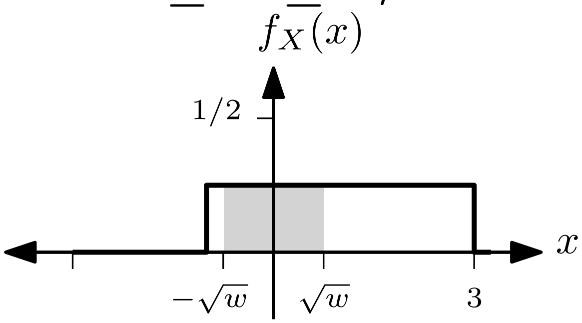


$$f_X(x) = \begin{cases} 1/4 & -1 \le x \le 3, \\ 0 & \text{otherwise.} \end{cases} \tag{6.11}$$

The integral (6.10) is somewhat tricky because the limits depend on the value of w. We first observe that $-1 \le X \le 3$ implies $0 \le W \le 9$. Thus $F_W(w) = 0$ for $w < 0$, and $F_W(w) = 1$ for $w > 9$. [Continued]

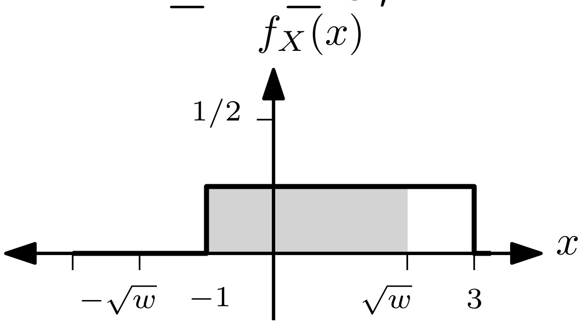
Example 6.4 Solution (Continued 3)

For $0 \leq w \leq 1$,



$$F_W(w) = \int_{-\sqrt{w}}^{\sqrt{w}} \frac{1}{4}\,dx = \frac{\sqrt{w}}{2}. \tag{6.12}$$

For $1 \leq w \leq 9$,

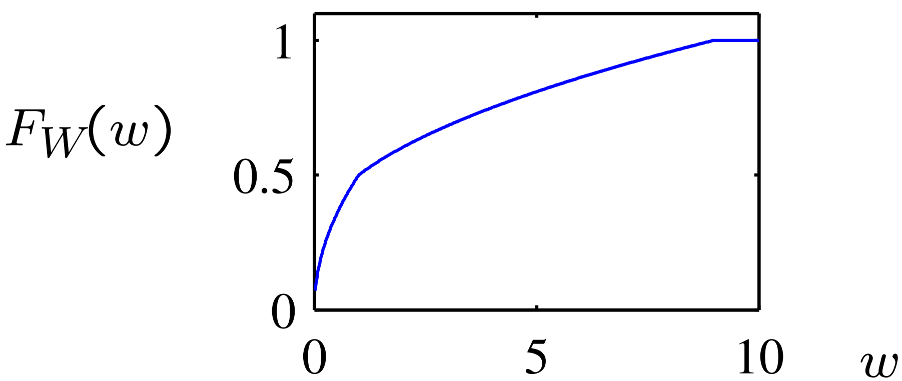


$$F_W(w) = \int_{-1}^{\sqrt{w}} \frac{1}{4}\,dx = \frac{\sqrt{w}+1}{4}. \tag{6.13}$$

[Continued]

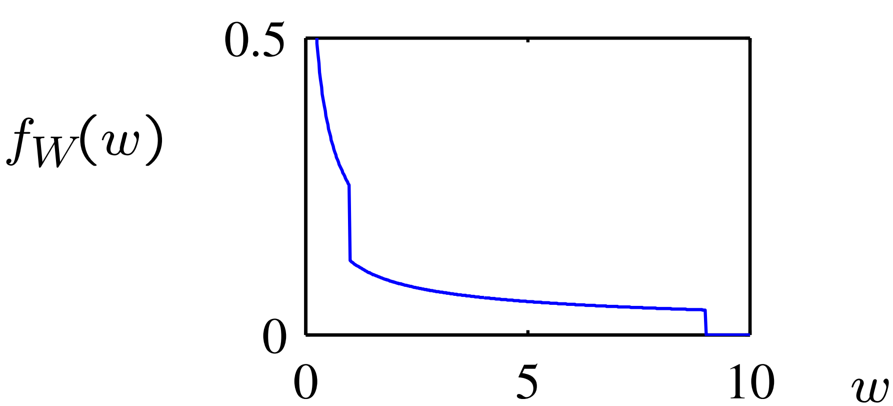
Example 6.4 Solution (Continued 4)

By combining the separate pieces, we can write a complete expression for $F_W(w)$:



$$F_W(w) = \begin{cases} 0 & w < 0, \\ \dfrac{\sqrt{w}}{2} & 0 \le w \le 1, \\ \dfrac{\sqrt{w}+1}{4} & 1 \le w \le 9, \\ 1 & w \ge 9. \end{cases} \tag{6.14}$$

To find $f_W(w)$, we take the derivative of $F_W(w)$ over each interval.



$$f_W(w) = \begin{cases} \dfrac{1}{4\sqrt{w}} & 0 \le w \le 1, \\ \dfrac{1}{8\sqrt{w}} & 1 \le w \le 9, \\ 0 & \text{otherwise.} \end{cases} \tag{6.15}$$

Theorem 6.5

Let U be a uniform $(0, 1)$ random variable and let $F(x)$ denote a cumulative distribution function with an inverse $F^{-1}(u)$ defined for $0 < u < 1$. The random variable $X = F^{-1}(U)$ has CDF $F_X(x) = F(x)$.

Proof: Theorem 6.5

First, we verify that $F^{-1}(u)$ is a nondecreasing function. To show this, suppose that for $u \geq u'$, $x = F^{-1}(u)$ and $x' = F^{-1}(u')$. In this case, $u = F(x)$ and $u' = F(x')$. Since $F(x)$ is nondecreasing, $F(x) \geq F(x')$ implies that $x \geq x'$. Hence, for the random variable $X = F^{-1}(U)$, we can write

$$F_X(x) = \mathsf{P}\left[F^{-1}(U) \leq x\right] = \mathsf{P}\left[U \leq F(x)\right] = F(x). \tag{6.16}$$

Example 6.5 Problem

U is the uniform $(0,1)$ random variable and $X = g(U)$. Derive $g(U)$ such that X is the exponential (1) random variable.

Example 6.5 Solution

The CDF of X is

$$F_X(x) = \begin{cases} 0 & x < 0, \\ 1 - e^{-x} & x \geq 0. \end{cases} \tag{6.17}$$

Note that if $u = F_X(x) = 1 - e^{-x}$, then $x = -\ln(1-u)$. That is, $F_X^{-1}(u) = -\ln(1-u)$ for $0 \leq u < 1$. Thus, by Theorem 6.5,

$$X = g(U) = -\ln(1-U) \tag{6.18}$$

is the exponential random variable with parameter $\lambda = 1$. Problem 6.2.7 asks the reader to derive the PDF of $X = -\ln(1-U)$ directly from first principles.

Example 6.6 Problem

For a uniform $(0, 1)$ random variable U, find a function $g(\cdot)$ such that $X = g(U)$ has a uniform (a, b) distribution.

Example 6.6 Solution

The CDF of X is

$$F_X(x) = \begin{cases} 0 & x < a, \\ (x-a)/(b-a) & a \leq x \leq b, \\ 1 & x > b. \end{cases} \tag{6.19}$$

For any u satisfying $0 \leq u \leq 1$, $u = F_X(x) = (x-a)/(b-a)$ if and only if

$$x = F_X^{-1}(u) = a + (b-a)u. \tag{6.20}$$

Thus by Theorem 6.5, $X = a+(b-a)U$ is a uniform (a, b) random variable. Note that we could have reached the same conclusion by observing that Theorem 6.3 implies $(b-a)U$ has a uniform $(0, b-a)$ distribution and that Theorem 6.4 implies $a + (b-a)U$ has a uniform $(a, (b-a)+a)$ distribution. Another approach, taken in Problem 6.2.11, is to derive the CDF and PDF of $a + (b-a)U$.

Quiz 6.2

X is an exponential (λ) PDF. Show that $Y = \sqrt{X}$ is a Rayleigh random variable (see Appendix A.2). Express the Rayleigh parameter a in terms of the exponential parameter λ.

Quiz 6.2 Solution

Since $Y = \sqrt{X}$, the fact that X is nonegative implies Y is non-negative. This implies $F_Y(y) = 0$ for $y < 0$. For $y \geq 0$, we find

$$\begin{aligned} F_Y(y) &= \mathrm{P}\left[\sqrt{X} \leq y\right] \\ &= \mathrm{P}\left[X \leq y^2\right] = F_X\left(y^2\right). \end{aligned} \tag{1}$$

For $x \geq 0$, $F_X(x) = 1 - e^{-\lambda x}$. Thus,

$$F_Y(y) = \begin{cases} 1 - e^{-\lambda y^2} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

By taking the derivative with respect to y, it follows that the PDF of Y is

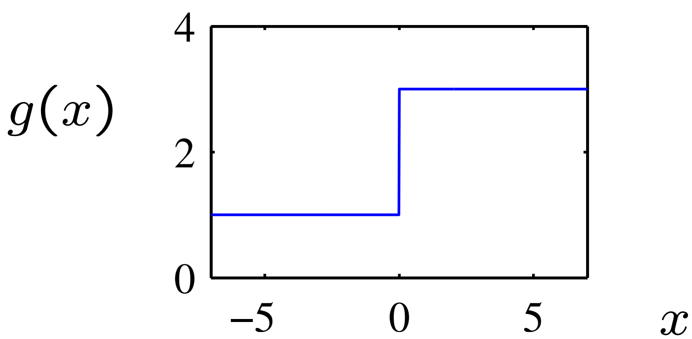
$$f_Y(y) = \begin{cases} 2\lambda y e^{-\lambda y^2} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

In comparing this result to the Rayleigh PDF given in Appendix A, we observe that Y is a Rayleigh (a) random variable with $a = \sqrt{2\lambda}$.

Functions Yielding Discrete or Mixed Random Variables

Example 6.7 Problem

Let X be a random variable with CDF $F_X(x)$. Let Y be the output of a clipping circuit, also referred to as a hard limiter, with the characteristic $Y = g(X)$ where



$$g(x) = \begin{cases} 1 & x \leq 0, \\ 3 & x > 0. \end{cases} \tag{6.21}$$

Express $F_Y(y)$ and $f_Y(y)$ in terms of $F_X(x)$ and $f_X(x)$.

Example 6.7 Solution

Before going deeply into the math, it is helpful to think about the nature of the derived random variable Y. The definition of $g(x)$ tells us that Y has only two possible values, $Y = 1$ and $Y = 3$. Thus Y is a discrete random variable. Furthermore, the CDF, $F_Y(y)$, has jumps at $y = 1$ and $y = 3$; it is zero for $y < 1$ and it is one for $y \geq 3$. Our job is to find the heights of the jumps at $y = 1$ and $y = 3$. In particular,

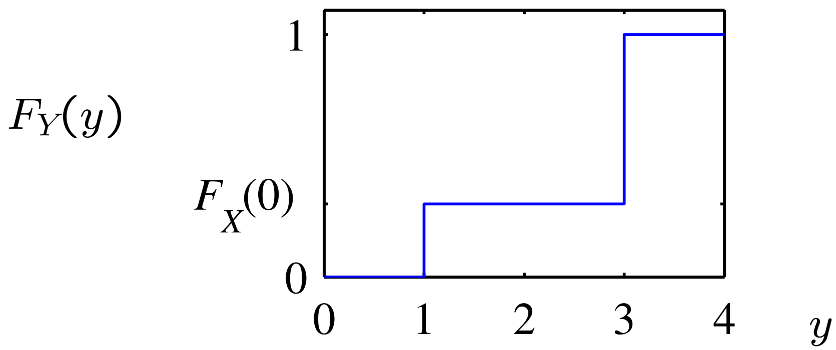
$$F_Y(1) = \text{P}[Y \leq 1] = \text{P}[X \leq 0] = F_X(0). \tag{6.22}$$

This tells us that the CDF jumps by $F_X(0)$ at $y = 1$. We also know that the CDF has to jump to one at $y = 3$.

[Continued]

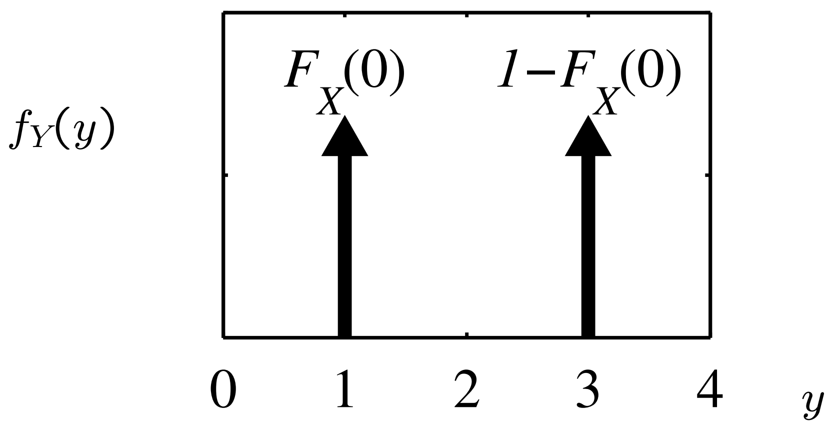
Example 6.7 Solution (Continued 2)

Therefore, the entire story is



$$F_Y(y) = \begin{cases} 0 & y < 1, \\ F_X(0) & 1 \le y < 3, \\ 1 & y \ge 3. \end{cases} \tag{6.23}$$

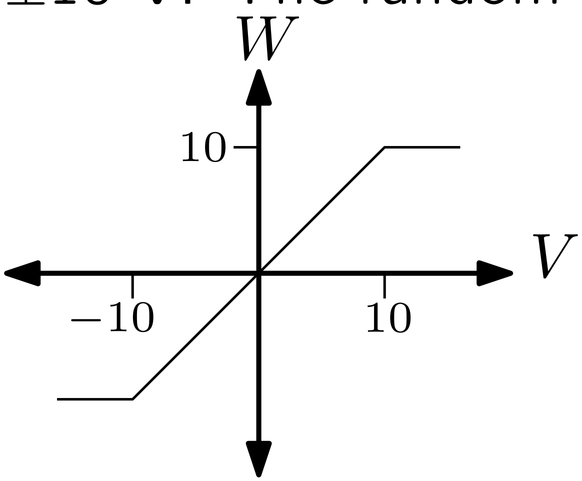
The PDF consists of impulses at $y = 1$ and $y = 3$. The weights of the impulses are the sizes of the two jumps in the CDF: $F_X(0)$ and $1 - F_X(0)$, respectively.



$$f_Y(y) = F_X(0)\,\delta(y-1) + [1 - F_X(0)]\delta(y-3).$$

Example 6.8 Problem

The output voltage of a microphone is a Gaussian random variable V with expected value $\mu_V = 0$ and standard deviation $\sigma_V = 5$ V. The microphone signal is the input to a soft limiter circuit with cutoff value ± 10 V. The random variable W is the output of the limiter:

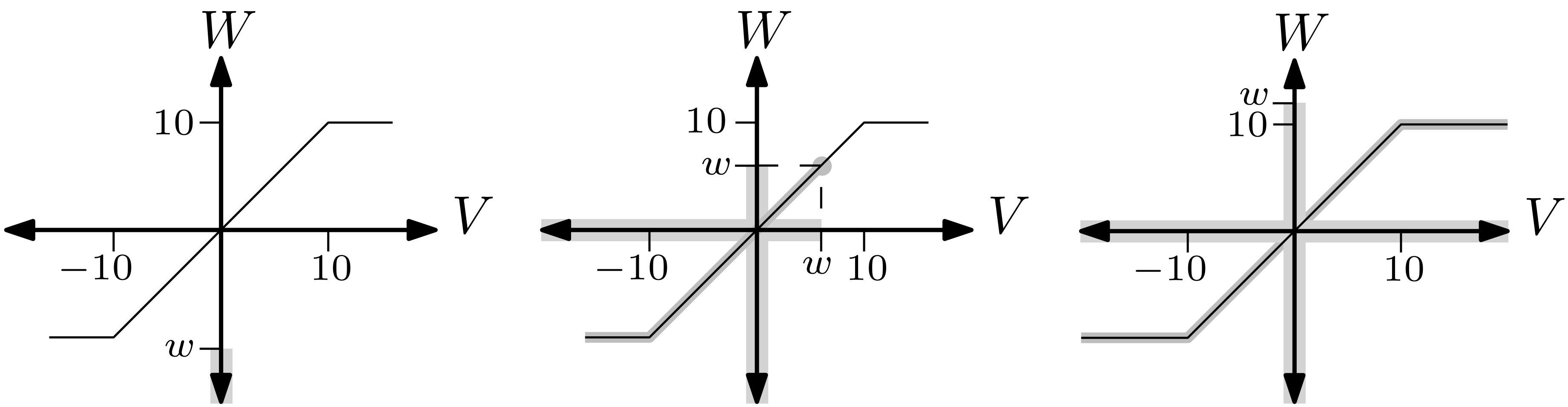


$$W = g(V) = \begin{cases} -10 & V < -10, \\ V & -10 \leq V \leq 10, \\ 10 & V > 10. \end{cases} \tag{6.24}$$

What are the CDF and PDF of W?

Example 6.8 Solution

To find the CDF, we need to find $F_W(w) = \mathrm{P}[W \leq w]$ for all values of w. The key is that all possible pairs (V, W) satisfy $W = g(V)$. This implies each w belongs to one of three cases:



(a) $w < -10$ **(b)** $-10 \leq w < 10$ **(c)** $w \geq 10$

(a) $w < -10$: From the function $W = g(V)$ we see that no possible pairs (V, W) satisfy $W \leq w < -10$. Hence $F_W(w) = \mathrm{P}[W \leq w] = 0$ in this case. This is perhaps a roundabout way of observing that $W = -10$ is the minumum possible W.

(b) $-10 \leq w < 10$: In this case we see that the event $\{W \leq w\}$, marked in gray on the vertical axis, corresponds to the event $\{V \leq w\}$, marked in gray on the horizontal axis. The corresponding (V, W) pairs are shown in the highlighted segment of the function $W = g(V)$. In this case, $F_W(w) = \mathrm{P}[W \leq w] = \mathrm{P}[V \leq w] = F_V(w)$. [Continued]

Example 6.8 Solution (Continued 2)

(c) $w \geq 10$: Here we see that the event $\{W \leq w\}$ corresponds to all values of V and $P[W \leq w] = P[V < \infty] = 1$. This is another way of saying $W = 10$ is the maximum W.

We combine these separate cases in the CDF

$$F_W(w) = P\left[W \leq w\right] = \begin{cases} 0 & w < -10, \\ F_V(w) & -10 \leq w < 10, \\ 1 & w \geq 10. \end{cases} \tag{6.25}$$

These conclusions are based solely on the structure of the limiter function $g(V)$ without regard for the probability model of V. Now we observe that because V is Gaussian $(0, 5)$, Theorem 4.14 states that $F_V(v) = \Phi(v/5)$. Therefore,

$$F_W(w) = \begin{cases} 0 & w < -10, \\ \Phi(w/5) & -10 \leq w \leq 10, \\ 1 & w > 10. \end{cases} \tag{6.26}$$

Note that the CDF jumps from 0 to $\Phi(-10/5) = 0.023$ at $w = -10$ and that it jumps from $\Phi(10/5) = 0.977$ to 1 at $w = 10$. Therefore,

$$f_W(w) = \frac{dF_W(w)}{dw} = \begin{cases} 0.023\delta(w+10) & w = -10, \\ \dfrac{1}{5\sqrt{2\pi}} e^{-w^2/50} & -10 < w < 10, \\ 0.023\delta(w-10) & w = 10, \\ 0 & \text{otherwise}. \end{cases} \tag{6.27}$$

Quiz 6.3

Random variable X is passed to a hard limiter that outputs Y. The PDF of X and the limiter output Y are

$$f_X(x) = \begin{cases} 1 - x/2 & 0 \leq x \leq 2, \\ 0 & \text{otherwise,} \end{cases} \qquad Y = \begin{cases} X & X \leq 1, \\ 1 & X > 1. \end{cases} \tag{6.28}$$

(a) What is the CDF $F_X(x)$?

(b) What is $P[Y = 1]$?

(c) What is $F_Y(y)$?

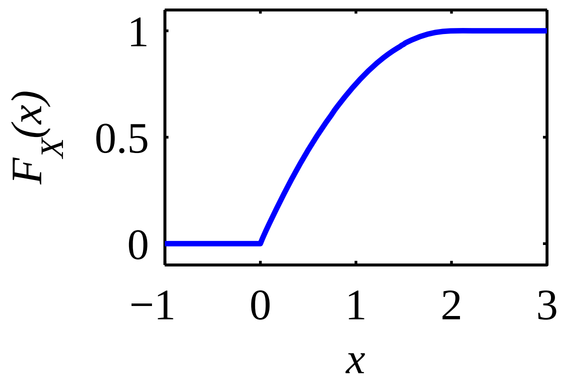
(d) What is $f_Y(y)$?

Quiz 6.3 Solution

(a) Since X is always nonnegative, $F_X(x) = 0$ for $x < 0$. Also, $F_X(x) = 1$ for $x \geq 2$ since its always true that $x \leq 2$. Lastly, for $0 \leq x \leq 2$,

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\, dy = \int_{0}^{x} (1 - y/2)\, dy = x - x^2/4. \tag{1}$$

The complete CDF of X is



$$F_X(x) = \begin{cases} 0 & x < 0, \\ x - x^2/4 & 0 \leq x \leq 2, \\ 1 & x > 2. \end{cases} \tag{2}$$

(b) The probability that $Y = 1$ is

$$\begin{aligned} P[Y = 1] &= P[X \geq 1] \\ &= 1 - F_X(1) = 1/4. \end{aligned} \tag{3}$$

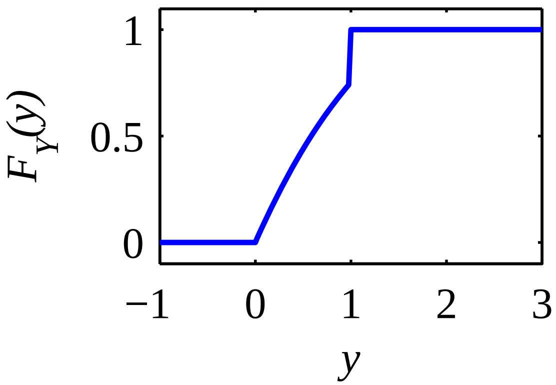
[Continued]

Quiz 6.3 Solution (Continued 2)

(c) Since X is nonnegative, Y is also nonnegative. Thus $F_Y(y) = 0$ for $y < 0$. Also, because $Y \leq 1$, $F_Y(y) = 1$ for all $y \geq 1$. Finally, for $0 < y < 1$,

$$\begin{aligned} F_Y(y) &= \mathrm{P}[Y \leq y] \\ &= \mathrm{P}[X \leq y] = F_X(y). \end{aligned} \tag{4}$$

Using the CDF $F_X(x)$, the complete expression for the CDF of Y is

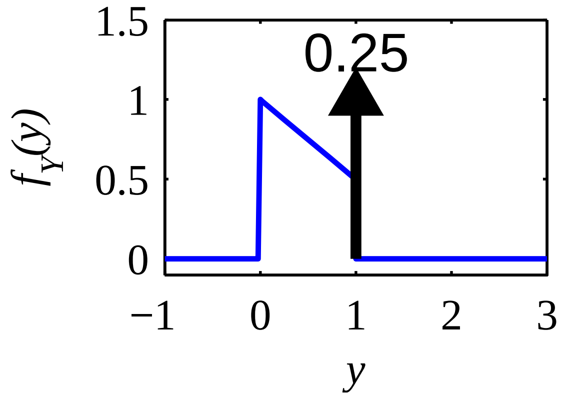


$$F_Y(y) = \begin{cases} 0 & y < 0, \\ y - y^2/4 & 0 \leq y < 1, \\ 1 & y \geq 1. \end{cases} \tag{5}$$

As expected, we see that the jump in $F_Y(y)$ at $y = 1$ is exactly equal to $\mathrm{P}[Y = 1]$.
[Continued]

Quiz 6.3 Solution (Continued 3)

(d) By taking the derivative of $F_Y(y)$, we obtain the PDF $f_Y(y)$. Note that when $y < 0$ or $y > 1$, the PDF is zero.



$$f_Y(y) = \begin{cases} 1 - \frac{y}{2} + \frac{\delta(y-1)}{4} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Section 6.4

Continuous Functions of Two Continuous Random Variables

Theorem 6.6

For continuous random variables X and Y, the CDF of $W = g(X, Y)$ is

$$F_W(w) = \mathrm{P}[W \leq w] = \iint_{g(x,y) \leq w} f_{X,Y}(x, y)\, dx\, dy.$$

Theorem 6.7

For continuous random variables X and Y, the CDF of $W = \max(X, Y)$ is

$$F_W(w) = F_{X,Y}(w, w) = \int_{-\infty}^{w} \int_{-\infty}^{w} f_{X,Y}(x, y)\, dx\, dy.$$

Example 6.9 Problem

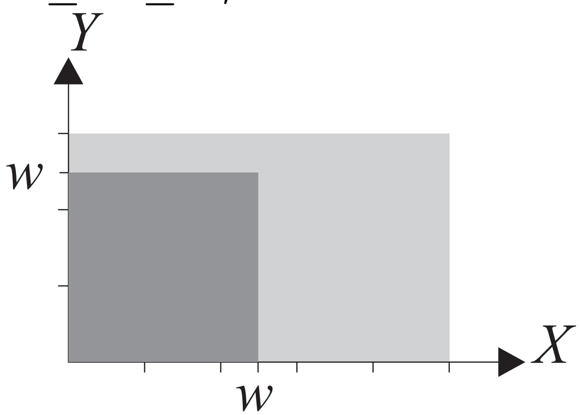
In Examples 5.7 and 5.9, X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/15 & 0 \le x \le 5, 0 \le y \le 3, \\ 0 & \text{otherwise.} \end{cases} \tag{6.29}$$

Find the PDF of $W = \max(X, Y)$.

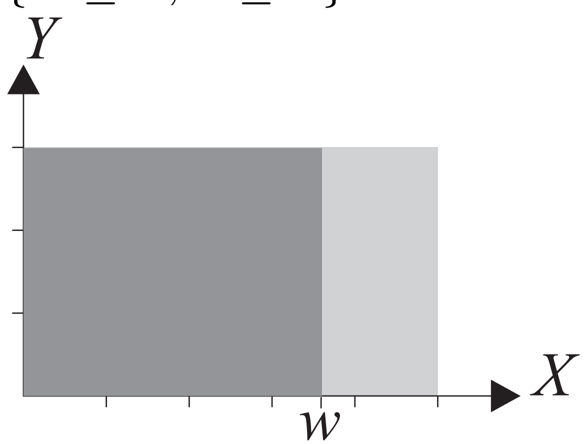
Example 6.9 Solution

Because $X \geq 0$ and $Y \geq 0$, $W \geq 0$. Therefore, $F_W(w) = 0$ for $w < 0$. Because $X \leq 5$ and $Y \leq 3$, $W \leq 5$. Thus $F_W(w) = 1$ for $w \geq 5$. For $0 \leq w \leq 5$, diagrams showing the regions of integration provide a guide to calculating $F_W(w)$. Two cases, $0 \leq w \leq 3$ and $3 \leq w \leq 5$, have to be considered separately. When $0 \leq w \leq 3$, Theorem 6.7 yields



$$F_W(w) = \int_0^w \int_0^w \frac{1}{15}\, dx\, dy = w^2/15. \tag{6.30}$$

Because the joint PDF is uniform, we see this probability is the area w^2 times the value of the joint PDF over that area. When $3 \leq w \leq 5$, the integral over the region $\{X \leq w, Y \leq w\}$ is



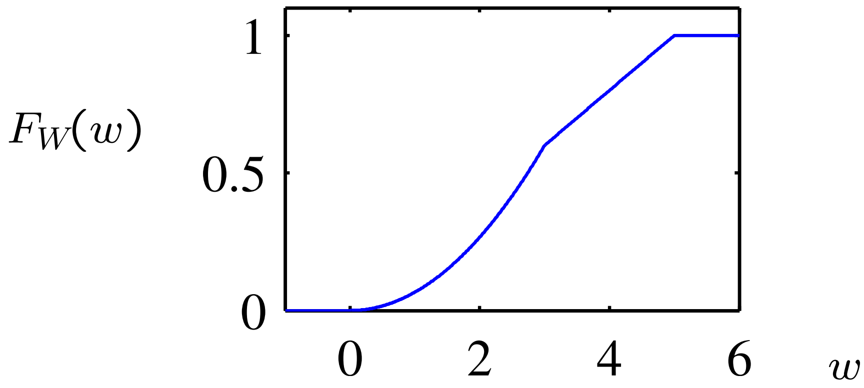
$$F_W(w) = \int_0^w \left(\int_0^3 \frac{1}{15}\, dy \right) dx = \int_0^w \frac{1}{5}\, dx = w/5, \tag{6.31}$$

which is the area $3w$ times the value of the joint PDF over that area.

[Continued]

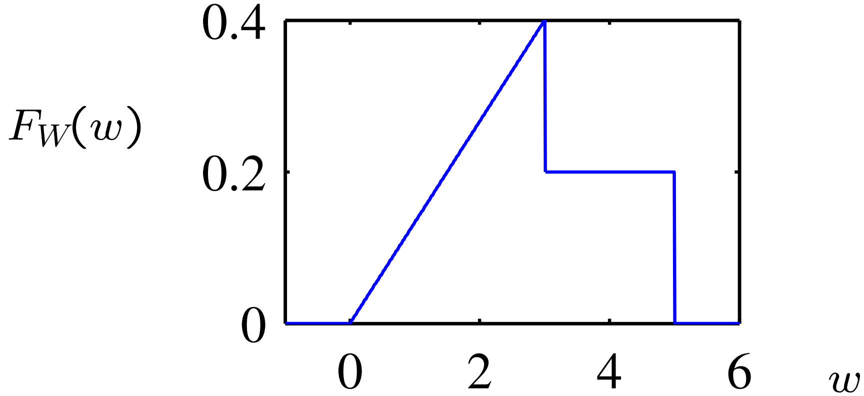
Example 6.9 Solution (Continued 2)

Combining the parts, we can write the joint CDF:



$$F_W(w) = \begin{cases} 0 & w < 0, \\ w^2/15 & 0 \le w \le 3, \\ w/5 & 3 < w \le 5, \\ 1 & w > 5. \end{cases} \qquad (6.32)$$

By taking the derivative, we find the corresponding joint PDF:



$$f_W(w) = \begin{cases} 2w/15 & 0 \le w \le 3, \\ 1/5 & 3 < w \le 5, \\ 0 & \text{otherwise.} \end{cases} \qquad (6.33)$$

Example 6.10 Problem

X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \lambda\mu e^{-(\lambda x+\mu y)} & x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6.34}$$

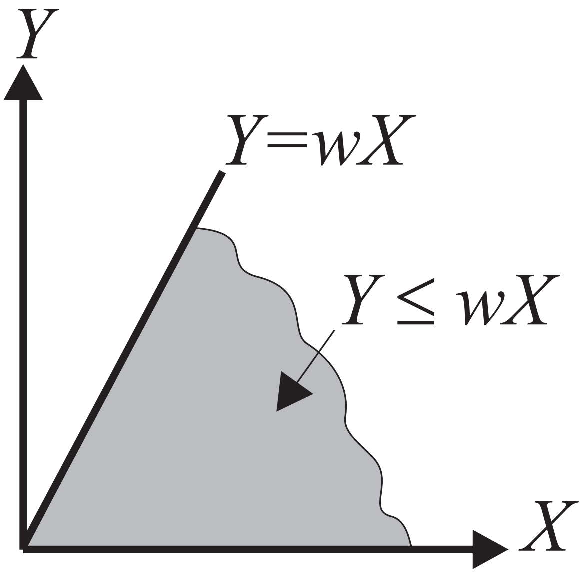
Find the PDF of $W = Y/X$.

Example 6.10 Solution

First we find the CDF:

$$F_W(w) = \text{P}[Y/X \leq w] = \text{P}[Y \leq wX]. \tag{6.35}$$

For $w < 0$, $F_W(w) = 0$. For $w \geq 0$, we integrate the joint PDF $f_{X,Y}(x,y)$ over the region of the X, Y plane for which $Y \leq wX$, $X \geq 0$, and $Y \geq 0$ as shown:



$$\begin{aligned} \text{P}[Y \leq wX] &= \int_0^\infty \left(\int_0^{wx} f_{X,Y}(x,y)\, dy \right) dx \\ &= \int_0^\infty \lambda e^{-\lambda x} \left(\int_0^{wx} \mu e^{-\mu y}\, dy \right) dx \\ &= \int_0^\infty \lambda e^{-\lambda x} \left(1 - e^{-\mu w x}\right) dx \\ &= 1 - \frac{\lambda}{\lambda + \mu w}. \end{aligned} \tag{6.36}$$

[Continued]

Example 6.10 Solution (Continued 2)

Therefore,

$$F_W(w) = \begin{cases} 0 & w < 0, \\ 1 - \dfrac{\lambda}{\lambda + \mu w} & \omega \geq 0. \end{cases} \tag{6.37}$$

Differentiating with respect to w, we obtain

$$f_W(w) = \begin{cases} \dfrac{\lambda\mu}{(\lambda + \mu w)^2} & w \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6.38}$$

Quiz 6.4(A)

A smartphone runs a news application that downloads Internet news every 15 minutes. At the start of a download, the radio modems negotiate a connection speed that depends on the radio channel quality. When the negotiated speed is low, the smartphone reduces the amount of news that it transfers to avoid wasting its battery. The number of kilobytes transmitted, L, and the speed B in kb/s, have the joint PMF

$P_{L,B}(l,b)$	$b = 512$	$b = 1,024$	$b = 2,048$
$l = 256$	0.2	0.1	0.05
$l = 768$	0.05	0.1	0.2
$l = 1536$	0	0.1	0.2

Let T denote the number of seconds needed for the transfer. Express T as a function of L and B. What is the PMF of T?

Quiz 6.4(A) Solution

The time required for the transfer is $T = 8L/B$. For each pair of values of L and B, we can calculate the time T needed for the transfer. We can write these down on the table for the joint PMF of L and B as follows:

$P_{L,B}(l,b)$	b=512	b=1024	b=2048
$l = 256$	0.20 (T=4)	0.10 (T=2)	0.05 (T=1)
$l = 768$	0.05 (T=12)	0.10 (T=6)	0.20 (T=3)
$l = 1536$	0.00 (T=24)	0.10 (T=12)	0.20 (T=6)

From the table, writing down the PMF of T is just bookkeeping. For example P$[T = 6] = 0.1 + 0.2 = 0.3$. The complete table of the PMF is

t	1	2	3	4	6	12
$P_T(t)$	0.05	0.1	0.2	0.2	0.3	0.15

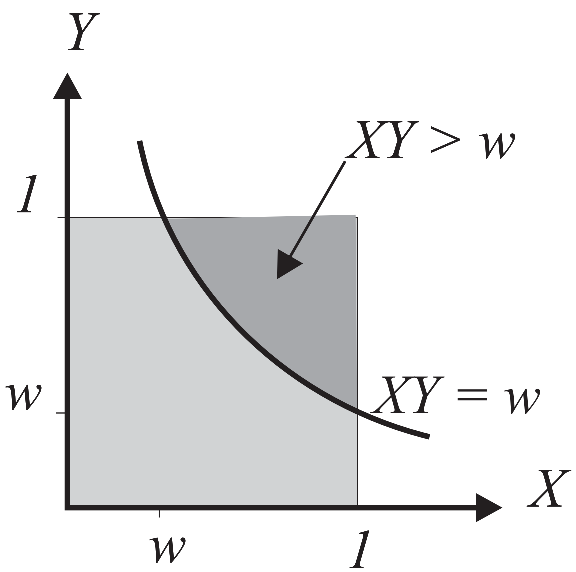
Quiz 6.4(B)

Find the CDF and the PDF of $W = XY$ when random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1 & 0 \le x \le 1, 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.39}$$

Quiz 6.4(B) Solution

First, we observe that since $0 \le X \le 1$ and $0 \le Y \le 1$, $W = XY$ satisfies $0 \le W \le 1$. Thus $f_W(0) = 0$ and $f_W(1) = 1$.



For $0 < w < 1$, we calculate the CDF $F_W(w) = \mathrm{P}[W \le w]$. As we see in the figure, the calculus is simpler if we integrate over the region $XY > w$. The calculus is

$$\begin{aligned} F_W(w) &= 1 - \mathrm{P}\left[XY > w\right] \\ &= 1 - \int_w^1 \int_{w/x}^1 dy\,dx \\ &= 1 - \int_w^1 (1 - w/x)\,dx \\ &= \left(x - w \ln x|_{x=w}^{x=1}\right) \\ &= w - w \ln w. \end{aligned} \qquad (1)$$

For $0 \le w \le 1$, the PDF is

$$f_W(w) = \frac{dF_W(w)}{dw} = -\ln w. \qquad (2)$$

The complete PDF of W is

$$f_W(w) = \begin{cases} 0 & w < 0, \\ -\ln w & 0 \le w \le 1, \\ 0 & w > 1. \end{cases} \qquad (3)$$

Section 6.5

PDF of the Sum of Two Random Variables

Theorem 6.8

The PDF of $W = X + Y$ is

$$f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, w - x)\ dx = \int_{-\infty}^{\infty} f_{X,Y}(w - y, y)\ dy.$$

Proof: Theorem 6.8

$$F_W(w) = \mathsf{P}\left[X + Y \leq w\right] = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{w-x} f_{X,Y}(x,y)\ dy \right) dx. \tag{6.40}$$

Taking the derivative of the CDF to find the PDF, we have

$$\begin{aligned} f_W(w) = \frac{dF_W(w)}{dw} &= \int_{-\infty}^{\infty} \left(\frac{d}{dw} \left(\int_{-\infty}^{w-x} f_{X,Y}(x,y)\ dy \right) \right) dx \\ &= \int_{-\infty}^{\infty} f_{X,Y}(x, w-x)\ dx. \end{aligned} \tag{6.41}$$

By making the substitution $y = w - x$, we obtain

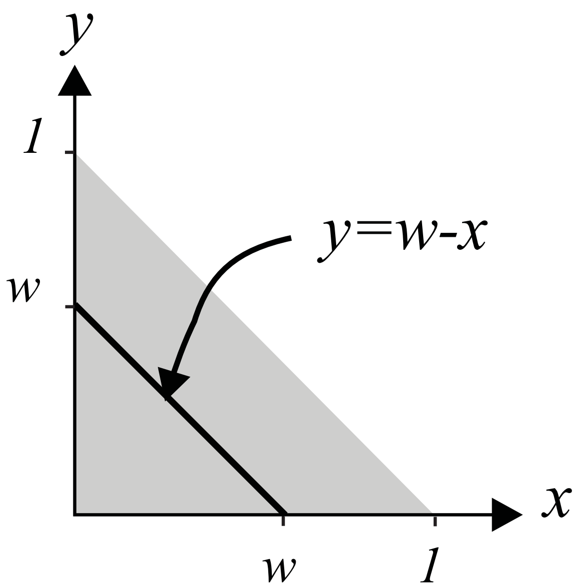
$$f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(w-y, y)\ dy. \tag{6.42}$$

Example 6.11 Problem

Find the PDF of $W = X + Y$ when X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le 1, 0 \le x \le 1, x + y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.43}$$

Example 6.11 Solution



The PDF of $W = X + Y$ can be found using Theorem 6.8. The possible values of X, Y are in the shaded triangular region where $0 \leq X + Y = W \leq 1$. Thus $f_W(w) = 0$ for $w < 0$ or $w > 1$. For $0 \leq w \leq 1$, applying Theorem 6.8 yields

$$f_W(w) = \int_0^w 2\,dx = 2w, \qquad 0 \leq w \leq 1. \tag{6.44}$$

The complete expression for the PDF of W is

$$f_W(w) = \begin{cases} 2w & 0 \leq w \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.45}$$

Theorem 6.9

When X and Y are independent random variables, the PDF of $W = X + Y$ is

$$f_W(w) = \int_{-\infty}^{\infty} f_X(w-y)\, f_Y(y)\ dy = \int_{-\infty}^{\infty} f_X(x)\, f_Y(w-x)\ dx.$$

Quiz 6.5

Let X and Y be independent exponential random variables with expected values $\mathrm{E}[X] = 1/3$ and $\mathrm{E}[Y] = 1/2$. Find the PDF of $W = X + Y$.

Quiz 6.5 Solution

Random variables X and Y have PDFs

$$f_X(x) = \begin{cases} 3e^{-3x} & x \geq 0, \\ 0 & \text{otherwise}, \end{cases} \qquad f_Y(y) = \begin{cases} 2e^{-2y} & y \geq 0, \\ 0 & \text{otherwise}. \end{cases} \tag{1}$$

Since X and Y are nonnegative, $W = X+Y$ is nonnegative and $f_W(w) = 0$ for $w < 0$. For $w > 0$, we use Theorem 6.9 to write

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_X(w-y)\, f_Y(y)\; dy \\ &= 6\int_0^w e^{-3(w-y)} e^{-2y}\, dy = 6e^{-3w}\int_0^w e^y\, dy = 6e^{-3w}\left(e^w - 1\right). \end{aligned} \tag{2}$$

The complete PDF of W is

$$f_W(w) = \begin{cases} 6(e^{-2w} - e^{-3w}) & w \geq 0, \\ 0 & \text{otherwise}. \end{cases}$$

Matlab

Example 6.12 Problem

Use Example 6.6 to write a Matlab function that generates m samples of a uniform (a, b) random variable.

Example 6.12 Solution

```
function x=uniformrv(a,b,m)
x=a+(b-a)*rand(m,1);
```

Example 6.6 says that $Y = a + (b - a)U$ is a uniform (a, b) random variable. We use this in `uniformrv`.

Example 6.13 Problem

Write a Matlab function that uses `icdfrv.m` to generate samples of Y, the maximum of three pointer spins, in Example 4.5.

Example 6.13 Solution

```
function y = icdf3spin(u);
y=u.^(1/3);
```

From Equation (4.18), we see that for $0 \leq y \leq 1$, $F_Y(y) = y^3$. If $u = F_Y(y) = y^3$, then $y = F_Y^{-1}(u) = u^{1/3}$. So we define (and save to disk) `icdf3spin.m`. Now, the function call `y=icdfrv(@icdf3spin,1000)` generates a vector holding 1000 samples of random variable Y. The notation `@icdf3spin` is the function handle for the function `icdf3spin.m`.

Quiz 6.6

Write a Matlab function `V=Vsample(m)` that returns m sample of random variable V with PDF

$$f_V(v) = \begin{cases} (v+5)/72 & -5 \le v \le 72, \\ 0 & \text{otherwise.} \end{cases} \tag{6.47}$$

Quiz 6.6 Solution

Your printing of this textbook may have typo. The PDF of V should be

$$f_V(v) = \begin{cases} (v+5)/72 & -5 \le v \le 7, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

First we find the corresponding CDF $F_V(v)$. For $-5 \le v \le 7$,

$$F_V(v) = \int_{-\infty}^{v} f_V(u)\, du = \int_{-5}^{v} \frac{u+5}{72}\, du \tag{2}$$

$$= \left.\frac{(u+5)^2}{144}\right|_{-5}^{v} = \frac{(v+5)^2}{144}. \tag{3}$$

The complete CDF of V is

$$F_V(v) = \begin{cases} 0 & v < -5, \\ (v+5)^2/144 & -5 \le v \le 7, \\ 1 & v > 7. \end{cases} \tag{4}$$

Now that we found the CDF $F_V(v)$, we can use Theorem 6.5. Over the interval $-5 \le v \le 7$, we find the inverse of the CDF by solving [Continued]

$$u = F_V(v) = \frac{(v+5)^2}{144} \tag{5}$$

for v as a function of u. This yields $v = 12\sqrt{u} - 5$. Thus, when U is a uniform $(0,1)$ random variable, the function

$$V = 12\sqrt{U} - 5 \tag{6}$$

generates samples of random variable V. In terms of Matlab, the code is simple:

```
function V = Vsample(m)
V=12*sqrt(rand(1,m))-5;
```

In `Vsample.m`, m samples of a uniform $(0,1)$ random variable are given by `rand(1,m)`. Here is a sample output

```
>> V=Vsample(5)
V =
    6.7402    3.3603    5.7350   -0.4799    2.7932
```