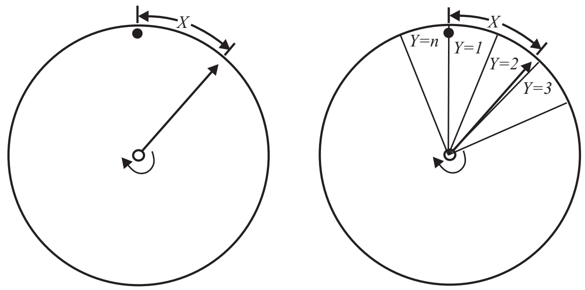
# Figure 4.1




The random pointer on disk of circumference 1.

# Example 4.1 Problem

Suppose we have a wheel of circumference one meter and we mark a point on the perimeter at the top of the wheel. In the center of the wheel is a radial pointer that we spin. After spinning the pointer, we measure the distance, $X$ meters, around the circumference of the wheel going clockwise from the marked point to the pointer position as shown in Figure 4.1. Clearly, $0 \leq X < 1$. Also, it is reasonable to believe that if the spin is hard enough, the pointer is just as likely to arrive at any part of the circle as at any other. For a given $x$, what is the probability $\mathrm{P}[X = x]$?

# Example 4.1 Solution

This problem is surprisingly difficult. However, given that we have developed methods for discrete random variables in Chapter 3, a reasonable approach is to find a discrete approximation to $X$. As shown on the right side of Figure 4.1, we can mark the perimeter with $n$ equal-length arcs numbered 1 to $n$ and let $Y$ denote the number of the arc in which the pointer stops. $Y$ is a discrete random variable with range $S_Y = \{1, 2, \ldots, n\}$. Since all parts of the wheel are equally likely, all arcs have the same probability. Thus the PMF of $Y$ is

$$P_Y(y) = \begin{cases} 1/n & y = 1, 2, \ldots, n, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

From the wheel on the right side of Figure 4.1, we can deduce that if $X = x$, then $Y = \lceil nx \rceil$, where the notation $\lceil a \rceil$ is defined as the smallest integer greater than or equal to $a$. Note that the event $\{X = x\} \subset \{Y = \lceil nx \rceil\}$, which implies that

$$\mathrm{P}[X = x] \leq \mathrm{P}[Y = \lceil nx \rceil] = \frac{1}{n}. \tag{4.2}$$

[Continued]

# Example 4.1 Solution (Continued 2)

We observe this is true no matter how finely we divide up the wheel. To find $\mathrm{P}[X=x]$, we consider larger and larger values of $n$. As $n$ increases, the arcs on the circle decrease in size, approaching a single point. The probability of the pointer arriving in any particular arc decreases until we have in the limit,

$$\mathrm{P}\left[X=x\right] \leq \lim_{n\to\infty} \mathrm{P}\left[Y=\lceil nx \rceil\right] = \lim_{n\to\infty} \frac{1}{n} = 0. \tag{4.3}$$

This demonstrates that $\mathrm{P}[X=x] \leq 0$. The first axiom of probability states that $\mathrm{P}[X=x] \geq 0$. Therefore, $\mathrm{P}[X=x]=0$. This is true regardless of the outcome, $x$. It follows that every outcome has probability zero.

# The Cumulative Distribution Function

## **Definition 4.1** Cumulative Distribution Function (CDF)

*The* cumulative distribution function *(CDF) of random variable* $X$ *is*

$$F_X(x) = \mathsf{P}\left[X \leq x\right].$$

# Theorem 4.1

For any random variable $X$,

(a) $F_X(-\infty) = 0$

(b) $F_X(\infty) = 1$

(c) $P[x_1 < X \leq x_2] = F_X(x_2) - F_X(x_1)$

# Definition 4.2 Continuous Random Variable

$X$ *is a* continuous random variable *if the CDF* $F_X(x)$ *is a continuous function.*

# Example 4.2 Problem

In the wheel-spinning experiment of Example 4.1, find the CDF of $X$.

# Example 4.2 Solution

We begin by observing that any outcome $x \in S_X = [0,1)$. This implies that $F_X(x) = 0$ for $x < 0$, and $F_X(x) = 1$ for $x \geq 1$. To find the CDF for $x$ between 0 and 1 we consider the event $\{X \leq x\}$, with $x$ growing from 0 to 1. Each event corresponds to an arc on the circle in Figure 4.1. The arc is small when $x \approx 0$ and it includes nearly the whole circle when $x \approx 1$. $F_X(x) = \text{P}[X \leq x]$ is the probability that the pointer stops somewhere in the arc. This probability grows from 0 to 1 as the arc increases to include the whole circle. Given our assumption that the pointer has no preferred stopping places, it is reasonable to expect the probability to grow in proportion to the fraction of the circle occupied by the arc $\{X \leq x\}$. This fraction is simply $x$. To be more formal, we can refer to Figure 4.1 and note that with the circle divided into $n$ arcs,

$$\{Y \leq \lceil nx \rceil - 1\} \subset \{X \leq x\} \subset \{Y \leq \lceil nx \rceil\}. \tag{4.4}$$

Therefore, the probabilities of the three events are related by

$$F_Y(\lceil nx \rceil - 1) \leq F_X(x) \leq F_Y(\lceil nx \rceil). \tag{4.5}$$

[Continued]

# Example 4.2 Solution (Continued 2)

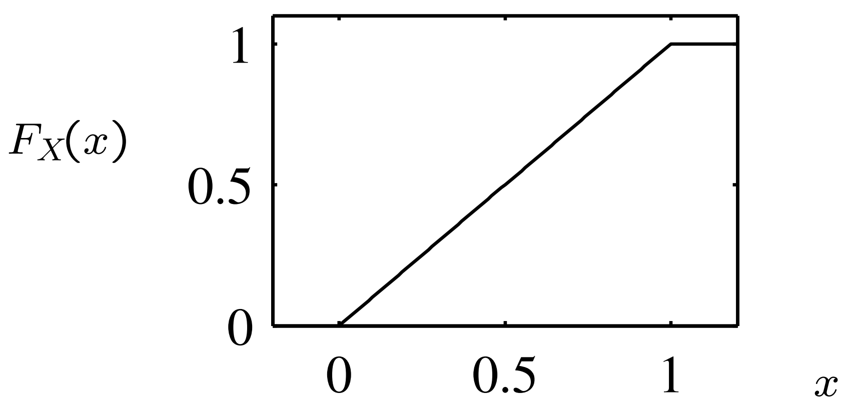
Note that $Y$ is a discrete random variable with CDF

$$F_Y(y) = \begin{cases} 0 & y < 0, \\ k/n & (k-1)/n < y \leq k/n, k = 1, 2, \ldots, n, \\ 1 & y > 1. \end{cases} \tag{4.6}$$

Thus for $x \in [0, 1)$ and for all $n$, we have

$$\frac{\lceil nx \rceil - 1}{n} \leq F_X(x) \leq \frac{\lceil nx \rceil}{n}. \tag{4.7}$$

In Problem 4.2.3, we ask the reader to verify that $\lim_{n\to\infty} \lceil nx \rceil / n = x$. This implies that as $n \to \infty$, both fractions approach $x$. The CDF of $X$ is




$$F_X(x) = \begin{cases} 0 & x < 0, \\ x & 0 \leq x < 1, \\ 1 & x \geq 1. \end{cases} \tag{4.8}$$

# Quiz 4.2

The cumulative distribution function of the random variable $Y$ is

$$F_Y(y) = \begin{cases} 0 & y < 0, \\ y/4 & 0 \le y \le 4, \\ 1 & y > 4. \end{cases} \tag{4.9}$$

Sketch the CDF of $Y$ and calculate the following probabilities:

(a) $P[Y \le -1]$

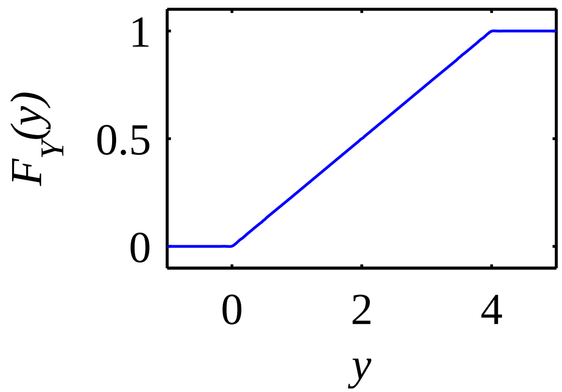
(b) $P[Y \le 1]$

(c) $P[2 < Y \le 3]$

(d) $P[Y > 1.5]$

# Quiz 4.2 Solution

The CDF of $Y$ is




$$F_Y(y) = \begin{cases} 0 & y < 0, \\ y/4 & 0 \le y \le 4, \\ 1 & y > 4. \end{cases} \tag{1}$$

From the CDF $F_Y(y)$, we can calculate the probabilities:

(a) $P[Y \le -1] = F_Y(-1) = 0$

(b) $P[Y \le 1] = F_Y(1) = 1/4$

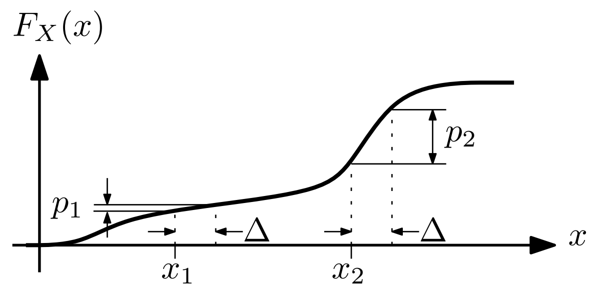
(c)
$$P[2 < Y \le 3] = F_Y(3) - F_Y(2) = 3/4 - 2/4 = 1/4.$$

(d)
$$\begin{aligned} P[Y > 1.5] &= 1 - P[Y \le 1.5] \\ &= 1 - F_Y(1.5) \\ &= 1 - (1.5)/4 = 5/8. \end{aligned}$$

# Section 4.3

## Probability Density Function

# Figure 4.2




The graph of an arbitrary CDF $F_X(x)$.

## Definition 4.3 (PDF)

*The* probability density function *(PDF) of a continuous random variable* $X$ *is*

$$f_X(x) = \frac{dF_X(x)}{dx}.$$

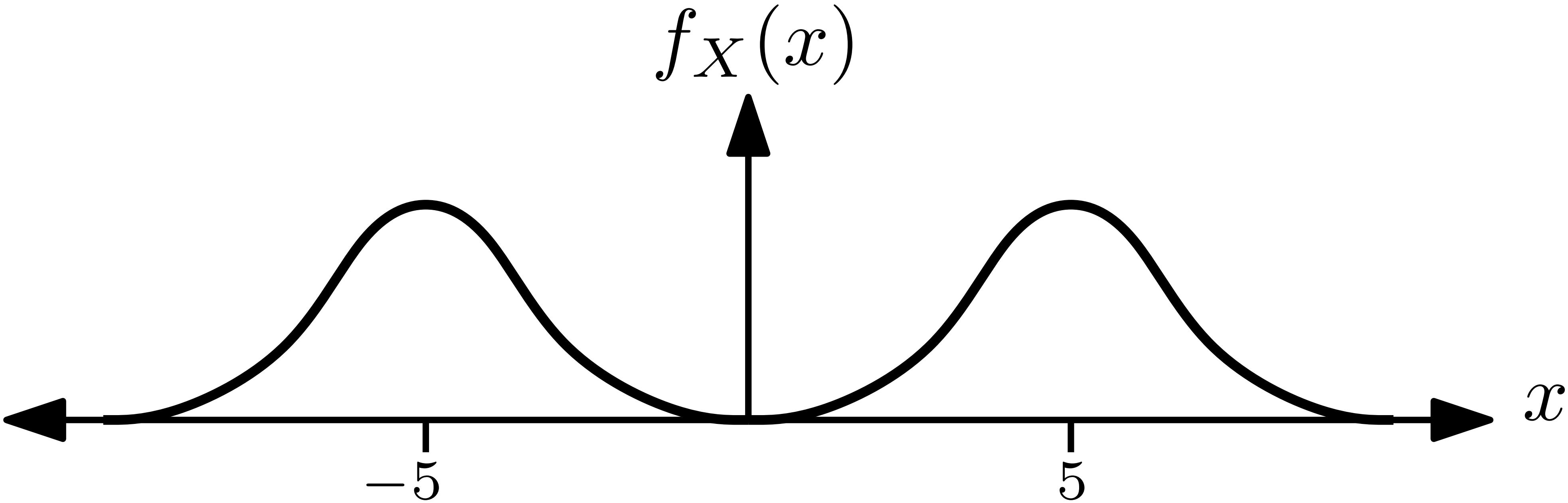
# Example 4.3 Problem

Figure 4.3 depicts the PDF of a random variable $X$ that describes the voltage at the receiver in a modem. What are probable values of $X$?

# Example 4.3 Solution

Note that there are two places where the PDF has high values and that it is low elsewhere. The PDF indicates that the random variable is likely to be near $-5$ V (corresponding to the symbol 0 transmitted) and near $+5$ V (corresponding to a 1 transmitted). Values far from $\pm 5$ V (due to strong distortion) are possible but much less likely.

# Figure 4.3




The PDF of the modem receiver voltage $X$.

# Theorem 4.2

For a continuous random variable $X$ with PDF $f_X(x)$,

(a) $f_X(x) \geq 0$ for all $x$,

(b) $F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$,

(c) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$.

# **Proof:** Theorem 4.2

The first statement is true because $F_X(x)$ is a nondecreasing function of $x$ and therefore its derivative, $f_X(x)$, is nonnegative. The second fact follows directly from the definition of $f_X(x)$ and the fact that $F_X(-\infty) = 0$. The third statement follows from the second one and Theorem 4.1(b).

# Theorem 4.3

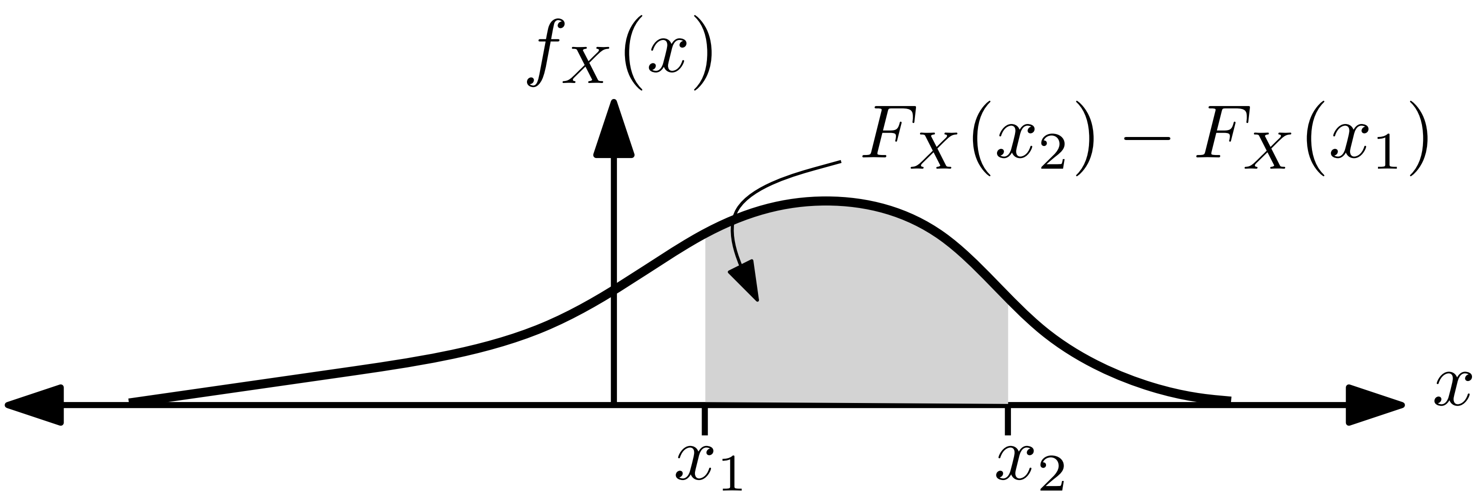
$$\mathsf{P}\left[x_1 < X \leq x_2\right] = \int_{x_1}^{x_2} f_X(x)\, dx.$$

# **Proof:** Theorem 4.3

From Theorem 4.1(c) and Theorem 4.2(b),

$$\begin{aligned} \mathrm{P}\left[x_1 < X \leq x_2\right] &= F_X(x_2) - F_X(x_1) \\ &= \int_{-\infty}^{x_2} f_X(x)\, dx - \int_{-\infty}^{x_1} f_X(x)\, dx = \int_{x_1}^{x_2} f_X(x)\, dx. \end{aligned} \tag{4.13}$$

# Figure 4.4



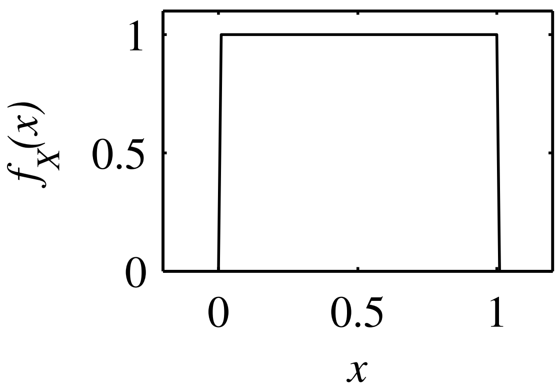

The PDF and CDF of $X$.

# Example 4.4 Problem

For the experiment in Examples 4.1 and 4.2, find the PDF of $X$ and the probability of the event $\{1/4 < X \leq 3/4\}$.

# Example 4.4 Solution

Taking the derivative of the CDF in Equation (4.8), $f_X(x) = 0$ when $x < 0$ or $x \geq 1$. For $x$ between 0 and 1 we have $f_X(x) = dF_X(x)/dx = 1$. Thus the PDF of $X$ is




$$f_X(x) = \begin{cases} 1 & 0 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.15}$$

The fact that the PDF is constant over the range of possible values of $X$ reflects the fact that the pointer has no favorite stopping places on the circumference of the circle. To find the probability that $X$ is between 1/4 and 3/4, we can use either Theorem 4.1 or Theorem 4.3. Thus

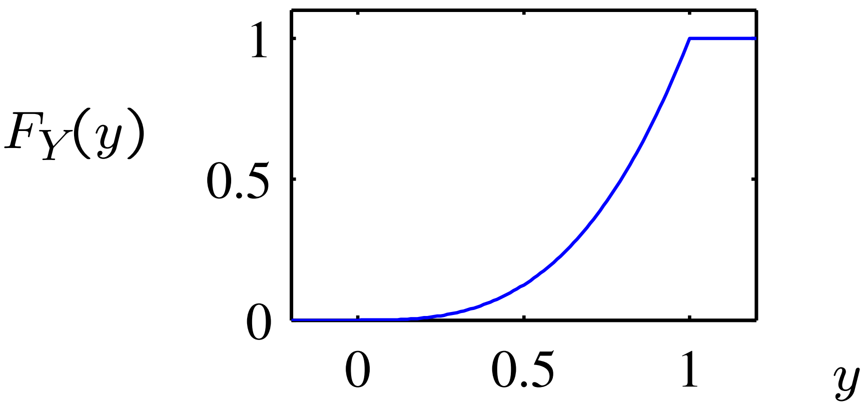
$$\mathrm{P}[1/4 < X \leq 3/4] = F_X(3/4) - F_X(1/4) = 1/2, \tag{4.16}$$

and equivalently,

$$\mathrm{P}[1/4 < X \leq 3/4] = \int_{1/4}^{3/4} f_X(x)\, dx = \int_{1/4}^{3/4} dx = 1/2. \tag{4.17}$$

# Example 4.5 Problem

Consider an experiment that consists of spinning the pointer in Example 4.1 three times and observing $Y$ meters, the maximum value of $X$ in the three spins. In Example 8.3, we show that the CDF of $Y$ is

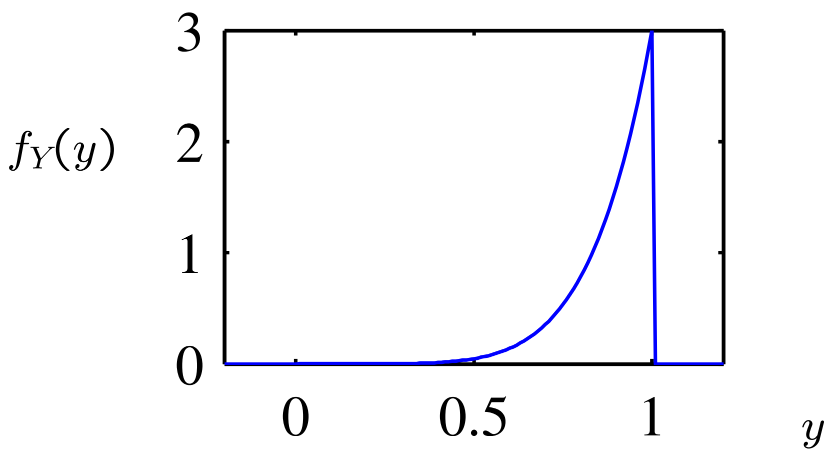



$$F_Y(y) = \begin{cases} 0 & y < 0, \\ y^3 & 0 \le y \le 1, \\ 1 & y > 1. \end{cases} \tag{4.18}$$

Find the PDF of $Y$ and the probability that $Y$ is between $1/4$ and $3/4$.

# Example 4.5 Solution

We apply Definition 4.3 to the CDF $F_Y(y)$. When $F_Y(y)$ is piecewise differentiable, we take the derivative of each piece:




$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} 3y^2 & 0 < y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.19}$$

Note that the PDF has values between 0 and 3. Its integral between any pair of numbers is less than or equal to 1. The graph of $f_Y(y)$ shows that there is a higher probability of finding $Y$ at the right side of the range of possible values than at the left side. This reflects the fact that the maximum of three spins produces higher numbers than individual spins. Either Theorem 4.1 or Theorem 4.3 can be used to calculate the probability of observing $Y$ between 1/4 and 3/4:

$$\mathrm{P}\left[1/4 < Y \leq 3/4\right] = F_Y(3/4) - F_Y(1/4) = (3/4)^3 - (1/4)^3 = 13/32, \tag{4.20}$$

and equivalently,

$$\mathrm{P}\left[1/4 < Y \leq 3/4\right] = \int_{1/4}^{3/4} f_Y(y)\, dy = \int_{1/4}^{3/4} 3y^2\, dy = 13/32. \tag{4.21}$$

Note that this probability is less than 1/2, which is the probability of $1/4 < X \leq 3/4$ calculated in Example 4.4 for one spin of the pointer.

# Quiz 4.3

Random variable $X$ has probability density function

$$f_X(x) = \begin{cases} cxe^{-x/2} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.23}$$

Sketch the PDF and find the following:

(a) the constant $c$

(b) the CDF $F_X(x)$

(c) $P[0 \leq X \leq 4]$

(d) $P[-2 \leq X \leq 2]$

# Quiz 4.3 Solution

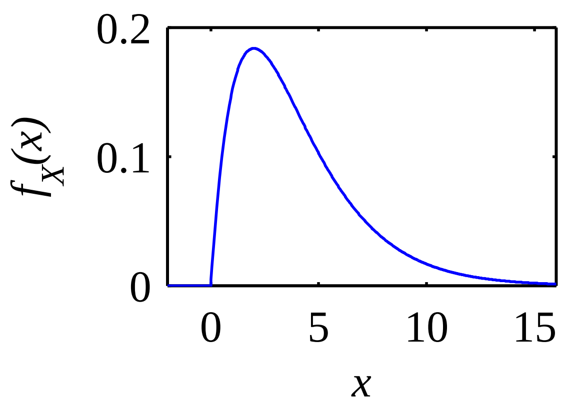
(a) First we will find the constant $c$ and then we will sketch the PDF. To find $c$, we use the fact that

$$1 = \int_{-\infty}^{\infty} f_X(x)\, dx = \int_0^{\infty} cxe^{-x/2}\, dx. \tag{1}$$

We evaluate this integral using integration by parts:

$$\begin{aligned} 1 &= \underbrace{-2cxe^{-x/2}\Big|_0^{\infty}}_{=0} + \int_0^{\infty} 2ce^{-x/2}\, dx \\ &= -4ce^{-x/2}\Big|_0^{\infty} = 4c. \end{aligned} \tag{2}$$

Thus $c = 1/4$ and $X$ has the Erlang $(n = 2, \lambda = 1/2)$ PDF



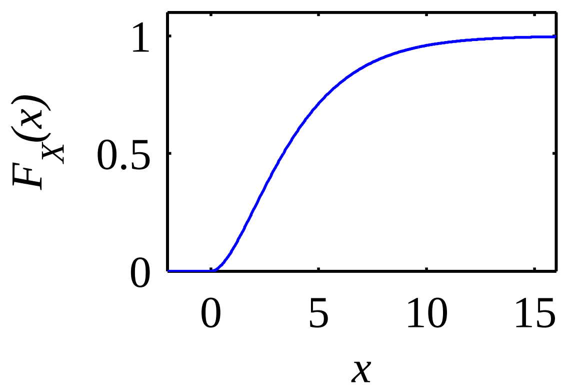

$$f_X(x) = \begin{cases} (x/4)e^{-x/2} & x \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

(b) To find the CDF $F_X(x)$, we first note $X$ is a nonnegative random variable so that $F_X(x) = 0$ for all $x < 0$. For $x \ge 0$,

[Continued]

$$F_X(x) = \int_0^x f_X(y)\, dy = \int_0^x \frac{y}{4} e^{-y/2}\, dy$$
$$= -\frac{y}{2} e^{-y/2}\Big|_0^x + \int_0^x \frac{1}{2} e^{-y/2}\, dy$$
$$= 1 - \frac{x}{2} e^{-x/2} - e^{-x/2}. \tag{3}$$

The complete expression for the CDF is




$$F_X(x) = \begin{cases} 1 - \left(\frac{x}{2} + 1\right) e^{-x/2} & x \geq 0, \\ 0 & \text{ow.} \end{cases}$$

(c) From the CDF $F_X(x)$,

$$P[0 \leq X \leq 4] = F_X(4) - F_X(0)$$
$$= 1 - 3e^{-2}. \tag{4}$$

(d) Similarly,

$$P[-2 \leq X \leq 2] = F_X(2) - F_X(-2)$$
$$= 1 - 3e^{-1}. \tag{5}$$

# Expected Values

# Definition 4.4 Expected Value

*The* expected value *of a continuous random variable* $X$ *is*

$$\mathsf{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

# Example 4.6 Problem

In Example 4.4, we found that the stopping point $X$ of the spinning wheel experiment was a uniform random variable with PDF

$$f_X(x) = \begin{cases} 1 & 0 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.28}$$

Find the expected stopping point $E[X]$ of the pointer.

# Example 4.6 Solution

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^1 x\, dx = 1/2 \text{ meter.} \tag{4.29}$$

With no preferred stopping points on the circle, the average stopping point of the pointer is exactly halfway around the circle.

# Example 4.7

Let $X$ be a uniform random variable with PDF

$$f_X(x) = \begin{cases} 1 & 0 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.30}$$

Let $W = g(X) = 0$ if $X \leq 1/2$, and $W = g(X) = 1$ if $X > 1/2$. $W$ is a discrete random variable with range $S_W = \{0, 1\}$.

# Theorem 4.4

The expected value of a function, $g(X)$, of random variable $X$ is

$$\mathsf{E}\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) f_X(x)\; dx.$$

# Theorem 4.5

For any random variable $X$,

(a) $\mathsf{E}[X - \mu_X] = 0$,

(b) $\mathsf{E}[aX + b] = a\,\mathsf{E}[X] + b$,

(c) $\mathsf{Var}[X] = \mathsf{E}[X^2] - \mu_X^2$,

(d) $\mathsf{Var}[aX + b] = a^2\,\mathsf{Var}[X]$.

# Example 4.8 Problem

Find the variance and standard deviation of the pointer position in Example 4.1.

# Example 4.8 Solution

To compute $\text{Var}[X]$, we use Theorem 4.5(c): $\text{Var}[X] = \text{E}[X^2] - \mu_X^2$. We calculate $\text{E}[X^2]$ directly from Theorem 4.4 with $g(X) = X^2$:

$$\text{E}\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^1 x^2\, dx = 1/3 \text{ m}^2. \tag{4.32}$$

In Example 4.6, we have $\text{E}[X] = 1/2$. Thus $\text{Var}[X] = 1/3 - (1/2)^2 = 1/12$, and the standard deviation is $\sigma_X = \sqrt{\text{Var}[X]} = 1/\sqrt{12} = 0.289$ meters.

# Quiz 4.4

The probability density function of the random variable $Y$ is

$$f_Y(y) = \begin{cases} 3y^2/2 & -1 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.33}$$

Sketch the PDF and find the following:

(a) the expected value $E[Y]$

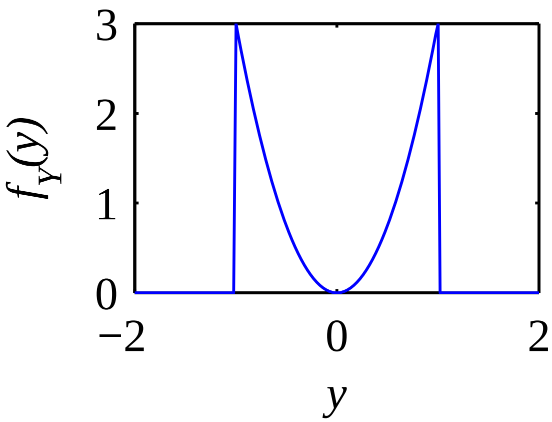
(b) the second moment $E[Y^2]$

(c) the variance $Var[Y]$

(d) the standard deviation $\sigma_Y$

# Quiz 4.4 Solution

The PDF of $Y$ is




$$f_Y(y) = \begin{cases} 3y^2/2 & -1 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) The expected value of $Y$ is

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_{-1}^{1} (3/2) y^3\, dy = (3/8) y^4 \Big|_{-1}^{1} = 0. \tag{2}$$

Note that the above calculation wasn't really necessary because $E[Y] = 0$ whenever the PDF $f_Y(y)$ is an even function, i.e., $f_Y(y) = f_Y(-y)$.

(b) The second moment of $Y$ is

$$E\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy = \int_{-1}^{1} (3/2) y^4\, dy = (3/10) y^5 \Big|_{-1}^{1} = 3/5. \tag{3}$$

(c) The variance of $Y$ is

$$\text{Var}[Y] = E\left[Y^2\right] - (E[Y])^2 = 3/5. \tag{4}$$

(d) The standard deviation of $Y$ is $\sigma_Y = \sqrt{\text{Var}[Y]} = \sqrt{3/5}$.

## Section 4.5

# Families of Continuous Random Variables

# Definition 4.5 Uniform Random Variable

*$X$ is a uniform $(a, b)$ random variable if the PDF of $X$ is*

$$f_X(x) = \begin{cases} 1/(b-a) & a \le x < b, \\ 0 & \text{otherwise}, \end{cases}$$

*where the two parameters are $b > a$.*

# Theorem 4.6

If $X$ is a uniform $(a, b)$ random variable,

- The CDF of $X$ is $$F_X(x) = \begin{cases} 0 & x \leq a, \\ (x-a)/(b-a) & a < x \leq b, \\ 1 & x > b. \end{cases}$$
- The expected value of $X$ is $E[X] = (b+a)/2$.
- The variance of $X$ is $\text{Var}[X] = (b-a)^2/12$.

# Example 4.9 Problem

The phase angle, $\Theta$, of the signal at the input to a modem is uniformly distributed between 0 and $2\pi$ radians. What are the PDF, CDF, expected value, and variance of $\Theta$?

# Example 4.9 Solution

From the problem statement, we identify the parameters of the uniform $(a, b)$ random variable as $a = 0$ and $b = 2\pi$. Therefore the PDF and CDF of $\Theta$ are

$$f_\Theta(\theta) = \begin{cases} 1/(2\pi) & 0 \le \theta < 2\pi, \\ 0 & \text{otherwise}, \end{cases} \quad F_\Theta(\theta) = \begin{cases} 0 & \theta \le 0, \\ \theta/(2\pi) & 0 < x \le 2\pi, \\ 1 & x > 2\pi. \end{cases} \tag{4.34}$$

The expected value is $E[\Theta] = b/2 = \pi$ radians, and the variance is $\text{Var}[\Theta] = (2\pi)^2/12 = \pi^2/3 \text{ rad}^2$.

# Theorem 4.7

Let $X$ be a uniform $(a, b)$ random variable, where $a$ and $b$ are both integers. Let $K = \lceil X \rceil$. Then $K$ is a discrete uniform $(a + 1, b)$ random variable.

# **Proof:** Theorem 4.7

Recall that for any $x$, $\lceil x \rceil$ is the smallest integer greater than or equal to $x$. It follows that the event $\{K = k\} = \{k - 1 < x \leq k\}$. Therefore,

$$\begin{aligned} \mathsf{P}[K = k] = P_K(k) &= \int_{k-1}^{k} P_X(x)\, dx \\ &= \begin{cases} 1/(b-a) & k = a+1, a+2, \ldots, b, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{4.35}$$

This expression for $P_K(k)$ conforms to Definition 3.8 of a discrete uniform $(a + 1, b)$ PMF.

# Definition 4.6 Exponential Random Variable

$X$ *is an* exponential ($\lambda$) random variable *if the PDF of* $X$ *is*

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & \textit{otherwise}, \end{cases}$$

*where the parameter* $\lambda > 0$.

# Example 4.10 Problem

The duration $T$ of a telephone call is often modeled as an exponential ($\lambda$) random variable.. If $\lambda = 1/3$, what is $\mathrm{E}[T]$, the expected duration of a telephone call? What are the variance and standard deviation of $T$? What is the probability that a call duration is within $\pm 1$ standard deviation of the expected call duration?

# Example 4.10 Solution

From Definition 4.6, $T$ has PDF

$$f_T(t) = \begin{cases} (1/3)e^{-t/3} & t \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.36}$$

With $f_T(t)$, we use Definition 4.4 to calculate the expected duration of a call:

$$\mathrm{E}[T] = \int_{-\infty}^{\infty} t f_T(t)\, dt = \int_0^{\infty} t \frac{1}{3} e^{-t/3}\, dt. \tag{4.37}$$

Integration by parts (Appendix B, Math Fact B.10) yields

$$\mathrm{E}[T] = -te^{-t/3}\Big|_0^{\infty} + \int_0^{\infty} e^{-t/3}\, dt = 3 \text{ minutes.} \tag{4.38}$$

To calculate the variance, we begin with the second moment of $T$:

$$\mathrm{E}\left[T^2\right] = \int_{-\infty}^{\infty} t^2 f_T(t)\, dt = \int_0^{\infty} t^2 \frac{1}{3} e^{-t/3}\, dt. \tag{4.39}$$

[Continued]

# Example 4.10 Solution (Continued 2)

Again integrating by parts, we have

$$\mathsf{E}\left[T^2\right] = -t^2 e^{-t/3}\Big|_0^\infty + \int_0^\infty (2t) e^{-t/3}\,dt = 2\int_0^\infty t e^{-t/3}\,dt. \tag{4.40}$$

With the knowledge that $\mathsf{E}[T] = 3$, we observe that $\int_0^\infty t e^{-t/3}\,dt = 3\,\mathsf{E}[T] = 9$. Thus $\mathsf{E}[T^2] = 6\,\mathsf{E}[T] = 18$ and

$$\mathsf{Var}[T] = \mathsf{E}\left[T^2\right] - (\mathsf{E}[T])^2 = 18 - 3^2 = 9 \text{ minutes}^2. \tag{4.41}$$

The standard deviation is $\sigma_T = \sqrt{\mathsf{Var}[T]} = 3$ minutes. The probability that the call duration is within 1 standard deviation of the expected value is

$$\mathsf{P}[0 \le T \le 6] = F_T(6) - F_T(0) = 1 - e^{-2} = 0.865 \tag{4.42}$$

# Theorem 4.8

If $X$ is an exponential ($\lambda$) random variable,

- The CDF of $X$ is $$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$
- The expected value of $X$ is $E[X] = 1/\lambda$.
- The variance of $X$ is $\text{Var}[X] = 1/\lambda^2$.

# Theorem 4.9

If $X$ is an exponential ($\lambda$) random variable, then $K = \lceil X \rceil$ is a geometric ($p$) random variable with $p = 1 - e^{-\lambda}$.

# **Proof:** Theorem 4.9

As in the Theorem 4.7 proof, the definition of $K$ implies

$$P_K(k) = \mathrm{P}[k-1 < X \leq k].$$

Referring to the CDF of $X$ in Theorem 4.8, we observe

$$P_K(k) = F_x(k) - F_x(k-1)$$
$$= \begin{cases} e^{-\lambda(k-1)} - e^{-\lambda k} & k = 1, 2, \ldots \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} (e^{-\lambda})^{k-1}(1 - e^{-\lambda}) & k = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases} \tag{4.43}$$

If we let $p = 1 - e^{-\lambda}$, we have

$$P_K(k) = \begin{cases} p(1-p)^{k-1} & k = 1, 2, \ldots \\ 0 & \text{otherwise,} \end{cases} \tag{4.44}$$

which conforms to Definition 3.5 of a geometric ($p$) random variable with $p = 1 - e^{-\lambda}$.

# Example 4.11 Problem

Phone company $A$ charges \$0.15 per minute for telephone calls. For any fraction of a minute at the end of a call, they charge for a full minute. Phone Company $B$ also charges \$0.15 per minute. However, Phone Company $B$ calculates its charge based on the exact duration of a call. If $T$, the duration of a call in minutes, is an exponential $(\lambda = 1/3)$ random variable, what are the expected revenues per call $\mathrm{E}[R_A]$ and $\mathrm{E}[R_B]$ for companies $A$ and $B$?

# Example 4.11 Solution

Because $T$ is an exponential random variable, we have in Theorem 4.8 (and in Example 4.10) $\mathsf{E}[T] = 1/\lambda = 3$ minutes per call. Therefore, for phone company $B$, which charges for the exact duration of a call,

$$\mathsf{E}[R_B] = 0.15\,\mathsf{E}[T] = \$0.45 \text{ per call.} \tag{4.45}$$

Company $A$, by contrast, collects $\$0.15\lceil T\rceil$ for a call of duration $T$ minutes. Theorem 4.9 states that $K = \lceil T\rceil$ is a geometric random variable with parameter $p = 1 - e^{-1/3}$. Therefore, the expected revenue for Company $A$ is

$$\mathsf{E}[R_A] = 0.15\,\mathsf{E}[K] = 0.15/p = (0.15)(3.53) = \$0.529 \text{ per call.} \tag{4.46}$$

# Definition 4.7 Erlang Random Variable

$X$ *is an* Erlang $(n, \lambda)$ *random variable if the PDF of* $X$ *is*

$$f_X(x) = \begin{cases} \dfrac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} & x \geq 0, \\ 0 & \textit{otherwise}, \end{cases}$$

*where the parameter* $\lambda > 0$, *and the parameter* $n \geq 1$ *is an integer.*

# Theorem 4.10

If $X$ is an Erlang $(n, \lambda)$ random variable, then

(a) $\mathrm{E}[X] = \dfrac{n}{\lambda}$,

(b) $\mathrm{Var}[X] = \dfrac{n}{\lambda^2}$.

# Theorem 4.11

Let $K_\alpha$ denote a Poisson $(\alpha)$ random variable. For any $x > 0$, the CDF of an Erlang $(n, \lambda)$ random variable $X$ satisfies

$$F_X(x) = 1 - F_{K_{\lambda x}}(n-1) = \begin{cases} 1 - \sum_{k=0}^{n-1} \frac{(\lambda x)^k e^{-\lambda x}}{k!}. & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

# Quiz 4.5

Continuous random variable $X$ has $\mathrm{E}[X] = 3$ and $\mathrm{Var}[X] = 9$. Find the PDF, $f_X(x)$, if

(a) $X$ is an exponential random variable,

(b) $X$ is a continuous uniform random variable.

(c) $X$ is an Erlang random variable.

# Quiz 4.5 Solution

(a) When $X$ is an exponential $(\lambda)$ random variable, $\mathrm{E}[X] = 1/\lambda$ and $\mathrm{Var}[X] = 1/\lambda^2$. Since $\mathrm{E}[X] = 3$ and $\mathrm{Var}[X] = 9$, we must have $\lambda = 1/3$. The PDF of $X$ is

$$f_X(x) = \begin{cases} (1/3)e^{-x/3} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(b) We know $X$ is a uniform $(a, b)$ random variable. To find $a$ and $b$, we apply Theorem 4.6 to write

$$\mathrm{E}[X] = \frac{a+b}{2} = 3 \tag{2}$$

$$\mathrm{Var}[X] = \frac{(b-a)^2}{12} = 9. \tag{3}$$

This implies

$$a + b = 6, \qquad b - a = \pm 6\sqrt{3}. \tag{4}$$

The only valid solution with $a < b$ is

$$a = 3 - 3\sqrt{3}, \qquad b = 3 + 3\sqrt{3}. \tag{5}$$

[Continued]

The complete expression for the PDF of $X$ is

$$f_X(x) = \begin{cases} 1/(6\sqrt{3}) & 3 - 3\sqrt{3} < x < 3 + 3\sqrt{3}, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

(c) We know that the Erlang $(n, \lambda)$ random variable has PDF

$$f_X(x) = \begin{cases} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

The expected value and variance are $\mathrm{E}[X] = n/\lambda$ and $\mathrm{Var}[X] = n/\lambda^2$. This implies

$$\frac{n}{\lambda} = 3, \qquad \frac{n}{\lambda^2} = 9. \tag{8}$$

It follows that

$$n = 3\lambda = 9\lambda^2. \tag{9}$$

Thus $\lambda = 1/3$ and $n = 1$. As a result, the Erlang $(n, \lambda)$ random variable must be the exponential $(\lambda = 1/3)$ random variable with PDF

$$f_X(x) = \begin{cases} (1/3)e^{-x/3} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

## Section 4.6

# Gaussian Random Variables

# Definition 4.8 Gaussian Random Variable

*$X$ is a Gaussian $(\mu, \sigma)$ random variable if the PDF of $X$ is*

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2},$$

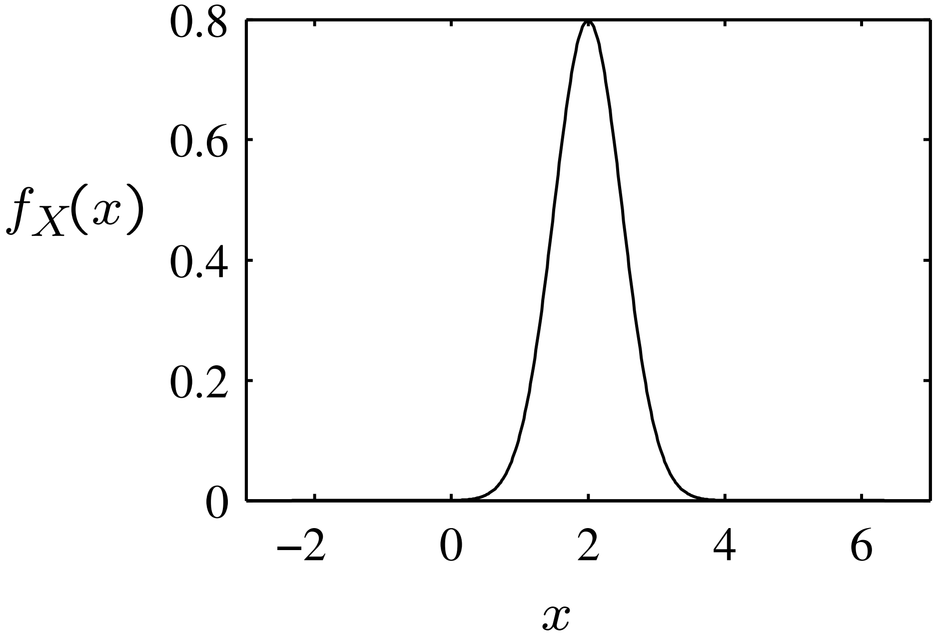
*where the parameter $\mu$ can be any real number and the parameter $\sigma > 0$.*

# Theorem 4.12

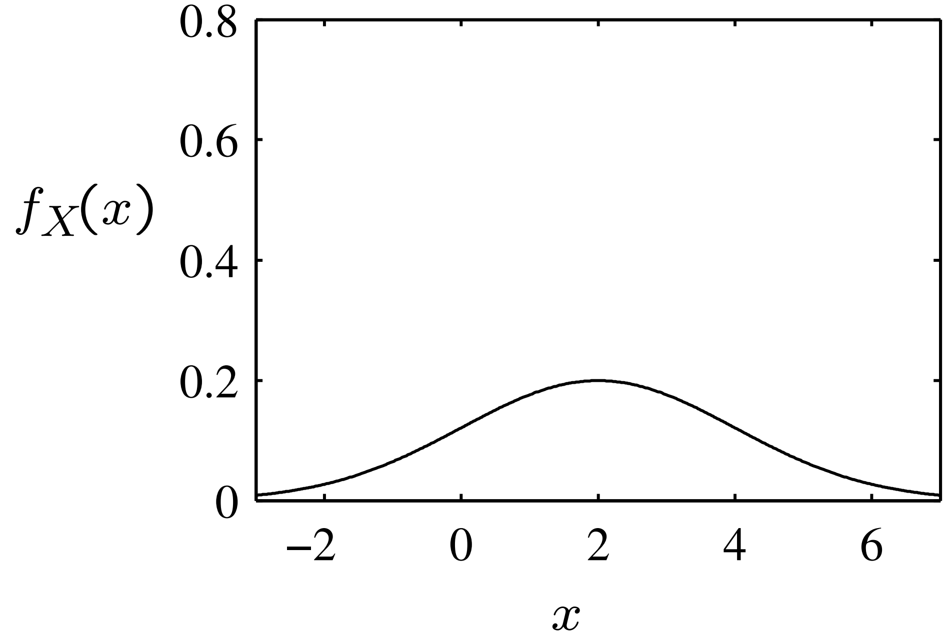
If $X$ is a Gaussian $(\mu, \sigma)$ random variable,

$$\mathrm{E}[X] = \mu \qquad \mathrm{Var}[X] = \sigma^2.$$

# Figure 4.5




**(a)** $\mu = 2$, $\sigma = 1/2$




**(b)** $\mu = 2$, $\sigma = 2$

Two examples of a Gaussian random variable $X$ with expected value $\mu$ and standard deviation $\sigma$.

# Theorem 4.13

If $X$ is Gaussian $(\mu, \sigma)$, $Y = aX + b$ is Gaussian $(a\mu + b, a\sigma)$.

## Definition 4.9 Standard Normal Random Variable

*The* standard normal random variable $Z$ *is the Gaussian* $(0, 1)$ *random variable.*

# Definition 4.10 Standard Normal CDF

*The CDF of the standard normal random variable* $Z$ *is*

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du.$$

# Theorem 4.14

If $X$ is a Gaussian $(\mu, \sigma)$ random variable, the CDF of $X$ is

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

The probability that $X$ is in the interval $(a, b]$ is

$$P[a < X \leq b] = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

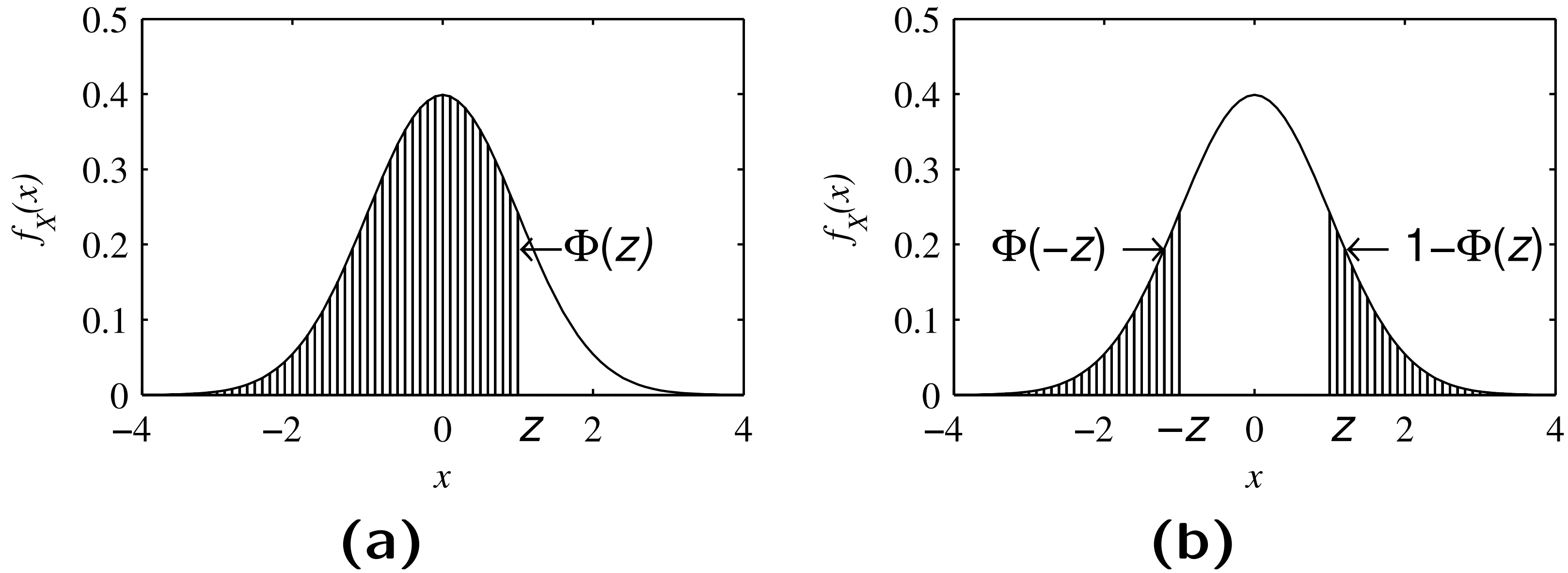
# Example 4.12 Problem

Suppose your score on a test is $x = 46$, a sample value of the Gaussian $(61, 10)$ random variable. Express your test score as a sample value of the standard normal random variable, $Z$.

# Example 4.12 Solution

Equation (4.47) indicates that $z = (46 - 61)/10 = -1.5$. Therefore your score is 1.5 *standard deviations less than the expected value*.

# Figure 4.6




Symmetry properties of the Gaussian $(0, 1)$ PDF.

# Theorem 4.15

$$\Phi(-z) = 1 - \Phi(z).$$

# Example 4.13 Problem

If $X$ is a Gaussian $(\mu = 61, \sigma = 10)$ random variable, what is $\mathrm{P}[51 < X \leq 71]$?

# Example 4.13 Solution

Applying Equation (4.47), $Z = (X - 61)/10$ and

$$\{51 < X \leq 71\} = \left\{-1 \leq \frac{X - 61}{10} \leq 1\right\} = \{-1 < Z \leq 1\}. \tag{4.48}$$

The probability of this event is

$$\begin{aligned} \mathrm{P}[-1 < Z \leq 1] &= \Phi(1) - \Phi(-1) \\ &= \Phi(1) - [1 - \Phi(1)] = 2\Phi(1) - 1 = 0.683. \end{aligned} \tag{4.49}$$

# Definition 4.11 Complementary CDF

*The* standard normal complementary CDF *is*

$$Q(z) = \mathrm{P}\left[Z > z\right] = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-u^2/2}\, du = 1 - \Phi(z).$$

# Example 4.14 Problem

In an optical fiber transmission system, the probability of a bit error is $Q(\sqrt{\gamma/2})$, where $\gamma$ is the signal-to-noise ratio. What is the minimum value of $\gamma$ that produces a bit error rate not exceeding $10^{-6}$?

# Example 4.14 Solution

Referring to Table 4.1, we find that $Q(z) < 10^{-6}$ when $z \geq 4.75$. Therefore, if $\sqrt{\gamma/2} \geq 4.75$, or $\gamma \geq 45$, the probability of error is less than $10^{-6}$. Although $10^(-6)$ seems a very small number, most practical optical fiber transmission systems have considerably lower binary error rates.

# Quiz 4.6

$X$ is the Gaussian $(0, 1)$ random variable and $Y$ is the Gaussian $(0, 2)$ random variable. Sketch the PDFs $f_X(x)$ and $f_Y(y)$ on the same axes and find:

(a) $P[-1 < X \leq 1]$,

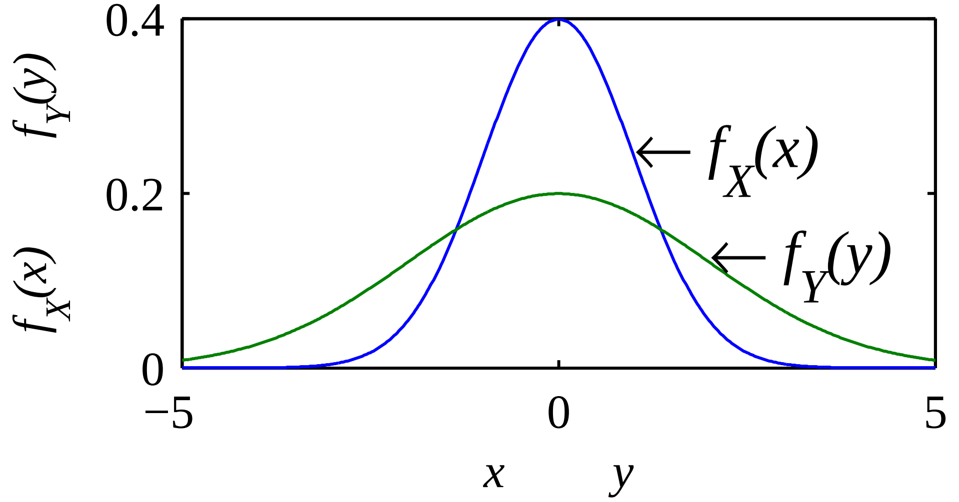
(b) $P[-1 < Y \leq 1]$,

(c) $P[X > 3.5]$,

(d) $P[Y > 3.5]$.

# Quiz 4.6 Solution

The PDFs of $X$ and $Y$ are:




The fact that $Y$ has twice the standard deviation of $X$ is reflected in the greater spread of $f_Y(y)$. However, it is important to remember that as the standard deviation increases, the peak value of the Gaussian PDF goes down.

Each of the requested probabilities can be calculated using $\Phi(z)$ function and Table 4.1 or $Q(z)$ and Table 4.2. [Continued]

# Quiz 4.6 Solution (Continued 2)

(a) Since $X$ is Gaussian $(0,1)$,

$$\begin{aligned} \mathrm{P}[-1 < X \le 1] &= F_X(1) - F_X(-1) \\ &= \Phi(1) - \Phi(-1) \\ &= 2\Phi(1) - 1 = 0.6826. \end{aligned} \tag{1}$$

(b) Since $Y$ is Gaussian $(0,2)$,

$$\begin{aligned} \mathrm{P}[-1 < Y \le 1] &= F_Y(1) - F_Y(-1) \\ &= \Phi\left(\frac{1}{\sigma_Y}\right) - \Phi\left(\frac{-1}{\sigma_Y}\right) \\ &= 2\Phi\left(\frac{1}{2}\right) - 1 = 0.383. \end{aligned} \tag{2}$$

(c) Again, since $X$ is Gaussian $(0,1)$, $\mathrm{P}[X > 3.5] = Q(3.5) = 2.33 \times 10^{-4}$.

(d) Since $Y$ is Gaussian $(0,2)$,

$$\mathrm{P}[Y > 3.5] = Q\left(\frac{3.5}{2}\right) = 1 - \Phi(1.75) = 0.04. \tag{3}$$

# Delta Functions, Mixed Random Variables

# **Definition 4.12** Unit Impulse (Delta) Function

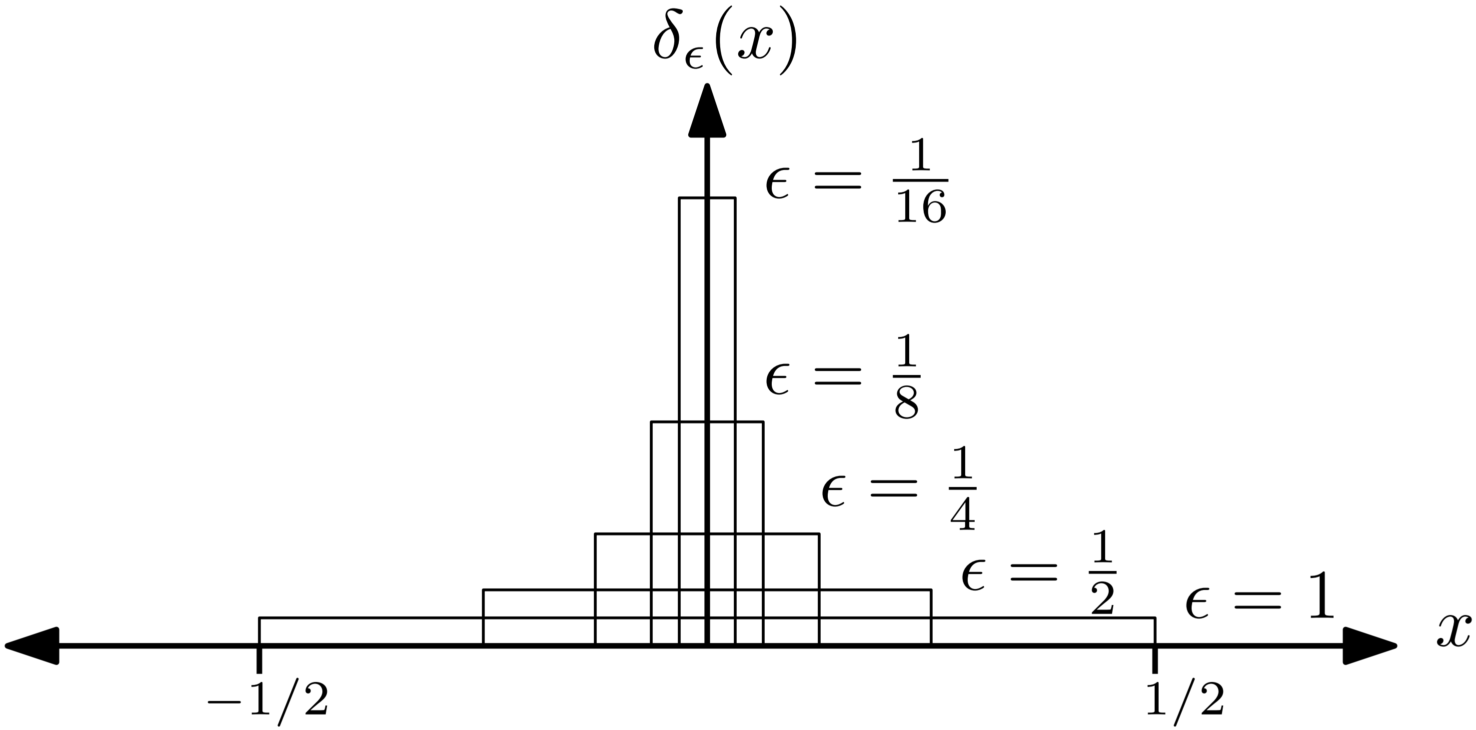
*Let*

$$d_\epsilon(x) = \begin{cases} 1/\epsilon & -\epsilon/2 \le x \le \epsilon/2, \\ 0 & \textit{otherwise.} \end{cases}$$

*The* unit impulse function *is*

$$\delta(x) = \lim_{\epsilon \to 0} d_\epsilon(x).$$

# Figure 4.7




As $\epsilon \to 0$, $d_\epsilon(x)$ approaches the delta function $\delta(x)$. For each $\epsilon$, the area under the curve of $d_\epsilon(x)$ equals 1.

# Theorem 4.16

For any continuous function $g(x)$,

$$\int_{-\infty}^{\infty} g(x)\delta(x - x_0)\, dx = g(x_0).$$

# **Definition 4.13** Unit Step Function

*The* unit step function *is*

$$u(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geq 0. \end{cases}$$

# Theorem 4.17

$$\int_{-\infty}^{x} \delta(v)\,dv = u(x).$$

# 4.7 Comment: CDF of a Discrete Random Variable

Consider the CDF of a discrete random variable, $X$. Recall that it is constant everywhere except at points $x_i \in S_X$, where it has jumps of height $P_X(x_i)$. Using the definition of the unit step function, we can write the CDF of $X$ as

$$F_X(x) = \sum_{x_i \in S_X} P_X(x_i)\, u(x - x_i). \tag{4.55}$$

From Definition 4.3, we take the derivative of $F_X(x)$ to find the PDF $f_X(x)$. Referring to Equation (4.54), the PDF of the discrete random variable $X$ is

$$f_X(x) = \sum_{x_i \in S_X} P_X(x_i)\, \delta(x - x_i). \tag{4.56}$$

# Example 4.15

Suppose $Y$ takes on the values $1, 2, 3$ with equal probability. The PMF and the corresponding CDF of $Y$ are

$$P_Y(y) = \begin{cases} 1/3 & y = 1, 2, 3, \\ 0 & \text{otherwise}, \end{cases} \qquad F_Y(y) = \begin{cases} 0 & y < 1, \\ 1/3 & 1 \le y < 2, \\ 2/3 & 2 \le y < 3, \\ 1 & y \ge 3. \end{cases} \tag{4.59}$$

Using the unit step function $u(y)$, we can write $F_Y(y)$ more compactly as

$$F_Y(y) = \frac{1}{3}u(y-1) + \frac{1}{3}u(y-2) + \frac{1}{3}u(y-3). \tag{4.60}$$

The PDF of $Y$ is

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{3}\delta(y-1) + \frac{1}{3}\delta(y-2) + \frac{1}{3}\delta(y-3). \tag{4.61}$$

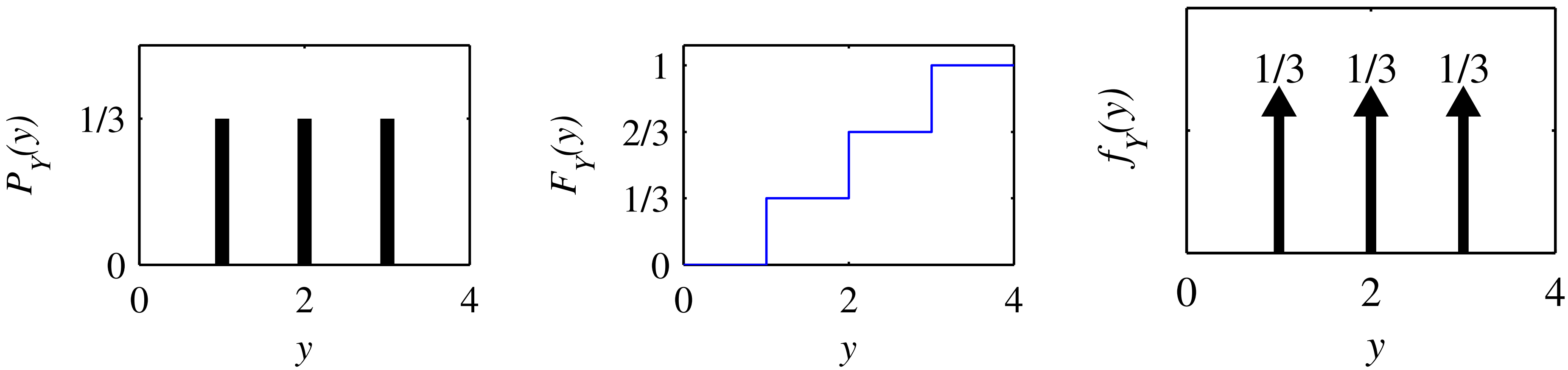
[Continued]

# Example 4.15 (Continued 2)

We see that the discrete random variable $Y$ can be represented graphically either by a PMF $P_Y(y)$ with bars at $y = 1, 2, 3$, by a CDF with jumps at $y = 1, 2, 3$, or by a PDF $f_Y(y)$ with impulses at $y = 1, 2, 3$. These three representations are shown in Figure 4.8. The expected value of $Y$ can be calculated either by summing over the PMF $P_Y(y)$ or integrating over the PDF $f_Y(y)$. Using the PDF, we have

$$\begin{aligned} \mathsf{E}[Y] &= \int_{-\infty}^{\infty} y f_Y(y)\, dy \\ &= \int_{-\infty}^{\infty} \frac{y}{3}\delta(y-1)\, dy + \int_{-\infty}^{\infty} \frac{y}{3}\delta(y-2)\, dy + \int_{-\infty}^{\infty} \frac{y}{3}\delta(y-3)\, dy \\ &= 1/3 + 2/3 + 1 = 2. \end{aligned} \tag{4.62}$$

# Figure 4.8




The PMF, CDF, and PDF of the discrete random variable $Y$.

# Example 4.16

For the random variable $Y$ of Example 4.15,

$$F_Y\left(2^-\right) = 1/3, \qquad F_Y\left(2^+\right) = 2/3. \tag{4.64}$$

# Theorem 4.18

For a random variable $X$, we have the following equivalent statements:

(a) $\mathrm{P}[X = x_0] = q$

(b) $P_X(x_0) = q$

(c) $F_X(x_0^+) - F_X(x_0^-) = q$

(d) $f_X(x_0) = q\delta(0)$

## **Definition 4.14** Mixed Random Variable

$X$ *is a* mixed *random variable if and only if* $f_X(x)$ *contains both impulses and nonzero, finite values.*

# Example 4.17 Problem

Observe someone dialing a telephone and record the duration of the call. In a simple model of the experiment, 1/3 of the calls never begin either because no one answers or the line is busy. The duration of these calls is 0 minutes. Otherwise, with probability 2/3, a call duration is uniformly distributed between 0 and 3 minutes. Let $Y$ denote the call duration. Find the CDF $F_Y(y)$, the PDF $f_Y(y)$, and the expected value $\mathrm{E}[Y]$.

# Example 4.17 Solution

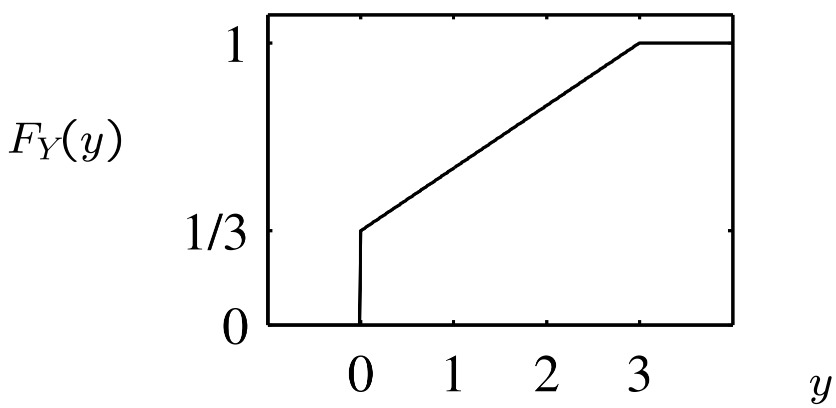
Let $A$ denote the event that the phone was answered. $\mathsf{P}[A] = 2/3$ and $\mathsf{P}[A^c] = 1/3$. Since $Y \geq 0$, we know that for $y < 0$, $F_Y(y) = 0$. Similarly, we know that for $y > 3$, $F_Y(y) = 1$. For $0 \leq y \leq 3$, we apply the law of total probability to write

$$F_Y(y) = \mathsf{P}[Y \leq y] = \mathsf{P}[Y \leq y|A^c]\,\mathsf{P}[A^c] + \mathsf{P}[Y \leq y|A]\,\mathsf{P}[A]. \tag{4.65}$$

When $A^c$ occurs, $Y = 0$, so that for $0 \leq y \leq 3$, $\mathsf{P}[Y \leq y|A^c] = 1$. When $A$ occurs, the call duration is uniformly distributed over $[0, 3]$, so that for $0 \leq y \leq 3$, $\mathsf{P}[Y \leq y|A] = y/3$. So, for $0 \leq y \leq 3$,

$$F_Y(y) = (1/3)(1) + (2/3)(y/3) = 1/3 + 2y/9. \tag{4.66}$$

The complete CDF of $Y$ is

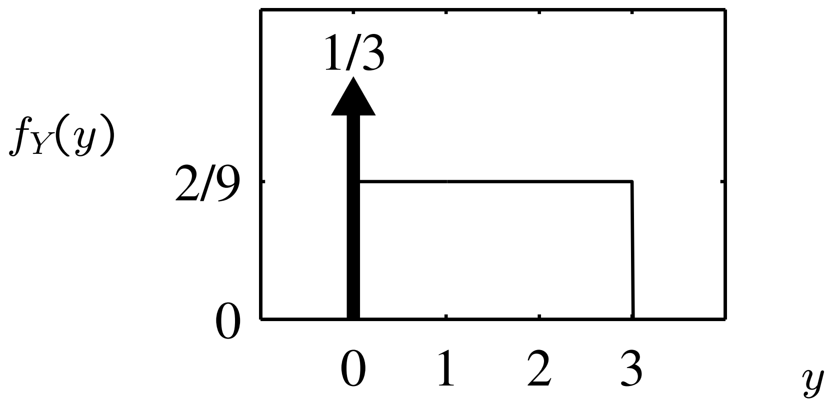



$$F_Y(y) = \begin{cases} 0 & y < 0, \\ 1/3 + 2y/9 & 0 \leq y < 3, \\ 1 & y \geq 3. \end{cases}$$

[Continued]

# Example 4.17 Solution (Continued 2)

Consequently, the corresponding PDF $f_Y(y)$ is




$$f_Y(y) = \begin{cases} \delta(y)/3 + 2/9 & 0 \le y \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

For the mixed random variable $Y$, it is easiest to calculate $\mathrm{E}[Y]$ using the PDF:

$$\mathrm{E}[Y] = \int_{-\infty}^{\infty} y \frac{1}{3} \delta(y)\, dy + \int_0^3 \frac{2}{9} y\, dy = 0 + \frac{2}{9} \frac{y^2}{2} \bigg|_0^3 = 1 \text{ minute.} \tag{4.67}$$

## 4.7 Comment: Variables

For any random variable $X$,

- $X$ always has a CDF $F_X(x) = \mathrm{P}[X \leq x]$.
- If $F_X(x)$ is piecewise flat with discontinuous jumps, then $X$ is discrete.
- If $F_X(x)$ is a continuous function, then $X$ is continuous.
- If $F_X(x)$ is a piecewise continuous function with discontinuities, then $X$ is mixed.
- When $X$ is discrete or mixed, the PDF $f_X(x)$ contains one or more delta functions.

# Quiz 4.7

The cumulative distribution function of random variable $X$ is

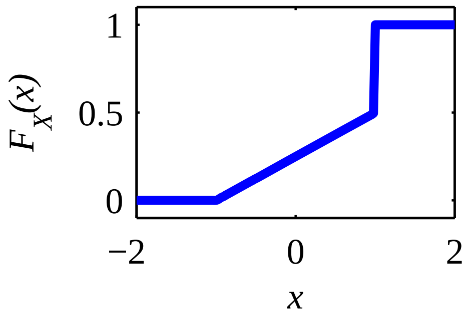
$$F_X(x) = \begin{cases} 0 & x < -1, \\ (x+1)/4 & -1 \le x < 1, \\ 1 & x \ge 1. \end{cases} \tag{4.68}$$

Sketch the CDF and find the following:

(a) $P[X \le 1]$
(b) $P[X < 1]$
(c) $P[X = 1]$
(d) the PDF $f_X(x)$

# Quiz 4.7 Solution

The CDF of $X$ is




$$F_X(x) = \begin{cases} 0 & x < -1, \\ (x+1)/4 & -1 \le x < 1, \\ 1 & x \ge 1. \end{cases} \tag{1}$$

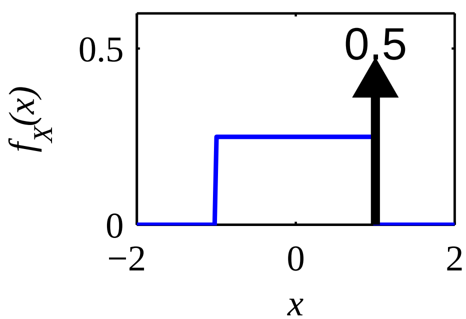
The following probabilities can be read directly from the CDF:

(a) $P[X \le 1] = F_X(1) = 1$.

(b) $P[X < 1] = F_X(1^-) = 1/2$.

(c) $P[X = 1] = F_X(1^+) - F_X(1^-) = 1/2$.

(d) We find the PDF $f_Y(y)$ by taking the derivative of $F_Y(y)$. The resulting PDF is




$$f_X(x) = \begin{cases} \frac{1}{4} & -1 \le x < 1, \\ \frac{\delta(x-1)}{2} & x = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

# Section 4.8

## Matlab

## 4.8 Comment: The Gaussian CDF in Matlab

For the Gaussian CDF, we use the built-in Matlab error function

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du. \tag{4.69}$$

It is related to the Gaussian CDF by

$$\Phi(x) = \frac{1}{2} + \frac{1}{2}\text{erf}\left(\frac{x}{\sqrt{2}}\right), \tag{4.70}$$

which is how we implement the Matlab function `phi(x)`.

# Matlab `rand`

- `y=rand(m,n)` is Matlab's approximation to a uniform $(0,1)$ random variable.
- It is an approximation for two reasons.
- First, `rand` produces pseudorandom numbers; the numbers seem random but are actually the output of a deterministic algorithm.
- Second, `rand` produces a double precision floating point number, represented in the computer by 64 bits.
- Thus Matlab distinguishes no more than $2^{64}$ unique double precision floating point numbers.
- By comparision, there are uncountably infinite real numbers in $[0,1)$.
- Even though `rand` is not random and does not have a continuous range, we can for all practical purposes use it as a source of independent sample values of the uniform $(0,1)$ random variable.

# Quiz 4.8

Use the Matlab function `uniformrv` and the result of Theorem 4.7 to write a Matlab function `y=duniform(a,b,m)` that produces $m$ samples of a discrete uniform $(a, b)$ random variable.

# Quiz 4.8 Solution

From Theorem 4.7, we know that if $X$ is a continuous uniform $(a-1, b)$ random variable, then $Y = \lceil X \rceil$ is a discrete uniform $(a, b)$ random variable. A Matlab function that implements this solution to produce $m$ samples of a discrete uniform $(a, b)$ random variable is:

```
function y = duniformrv(a,b,m)
x=uniformrv(a-1,b,m);
y=ceil(x);
```

Note that `ceil(x)` is the Matlab implementation of $\lceil x \rceil$.