HW2-Sol

Problem 3.3.17

(a) Since each day is independent of any other day, $P[W_{33}]$ is just the probability that a winning lottery ticket was bought. Similarly for $P[L_{87}]$ and $P[N_{99}]$ become just the probability that a losing ticket was bought and that no ticket was bought on a single day, respectively. Therefore

$$P[W_{33}] = p/2, \qquad P[L_{87}] = (1-p)/2, \qquad P[N_{99}] = 1/2. \tag{1}$$

(b) Suppose we say a success occurs on the kth trial if on day k we buy a ticket. Otherwise, a failure occurs. The probability of success is simply 1/2. The random variable K is just the number of trials until the first success and has the geometric PMF

$$P_K(k) = \begin{cases} (1/2)(1/2)^{k-1} = (1/2)^k & k = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(c) The probability that you decide to buy a ticket and it is a losing ticket is $(1-p)/2$, independent of any other day. If we view buying a losing ticket as a Bernoulli success, R, the number of losing lottery tickets bought in m days, has the binomial PMF

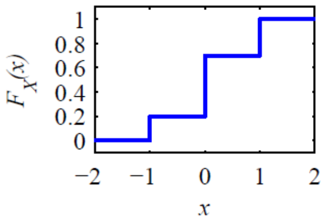
$$P_R(r) = \begin{cases} \binom{m}{r}[(1-p)/2]^r[(1+p)/2]^{m-r} & r = 0, 1, \ldots, m, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

(d) Letting D be the day on which the j-th losing ticket is bought, we can find the probability that $D = d$ by noting that $j-1$ losing tickets must have been purchased in the $d-1$ previous days. Therefore D has the Pascal PMF

$$P_D(d) = \begin{cases} \binom{d-1}{j-1}[(1-p)/2]^j[(1+p)/2]^{d-j} & d = j, j+1, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Problem 3.4.2

(a) The given CDF is shown in the diagram below.



$$F_X(x) = \begin{cases} 0 & x < -1, \\ 0.2 & -1 \le x < 0, \\ 0.7 & 0 \le x < 1, \\ 1 & x \ge 1. \end{cases} \tag{1}$$

(b) The corresponding PMF of X is

$$P_X(x) = \begin{cases} 0.2 & x = -1, \\ 0.5 & x = 0, \\ 0.3 & x = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Problem 3.5.17

We write the sum as a double sum in the following way:

$$\sum_{i=0}^{\infty} \mathrm{P}[X > i] = \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} P_X(j). \tag{1}$$

At this point, the key step is to reverse the order of summation. You may need to make a sketch of the feasible values for i and j to see how this reversal occurs. In this case,

$$\sum_{i=0}^{\infty} \mathrm{P}[X > i] = \sum_{j=1}^{\infty} \sum_{i=0}^{j-1} P_X(j) = \sum_{j=1}^{\infty} j P_X(j) = \mathrm{E}[X]. \tag{2}$$

Problem 3.6.5

The cellular calling plan charges a flat rate of \$20 per month up to and including the 30th minute, and an additional 50 cents for each minute over 30 minutes. Knowing that the time you spend on the phone is a geometric random variable M with mean $1/p = 30$, the PMF of M is

$$P_M(m) = \begin{cases} (1-p)^{m-1}p & m = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The monthly cost, C obeys

$$P_C(20) = \mathrm{P}[M \leq 30] = \sum_{m=1}^{30} (1-p)^{m-1}p = 1 - (1-p)^{30}. \tag{2}$$

When $M \geq 30$, $C = 20 + (M - 30)/2$ or $M = 2C - 10$. Thus,

$$P_C(c) = P_M(2c - 10) \qquad c = 20.5, 21, 21.5, \ldots. \tag{3}$$

The complete PMF of C is

$$P_C(c) = \begin{cases} 1 - (1-p)^{30} & c = 20, \\ (1-p)^{2c-10-1}p & c = 20.5, 21, 21.5, \ldots. \end{cases} \tag{4}$$

Problem 3.7.7

As a function of the number of minutes used, M, the monthly cost is

$$C(M) = \begin{cases} 20 & M \leq 30 \\ 20 + (M-30)/2 & M \geq 30 \end{cases} \tag{1}$$

The expected cost per month is

$$\begin{aligned} \mathrm{E}[C] &= \sum_{m=1}^{\infty} C(m) P_M(m) \\ &= \sum_{m=1}^{30} 20 P_M(m) + \sum_{m=31}^{\infty} (20 + (m-30)/2) P_M(m) \\ &= 20 \sum_{m=1}^{\infty} P_M(m) + \frac{1}{2} \sum_{m=31}^{\infty} (m-30) P_M(m). \end{aligned} \tag{2}$$

Since $\sum_{m=1}^{\infty} P_M(m) = 1$ and since $P_M(m) = (1-p)^{m-1} p$ for $m \geq 1$, we have

$$\mathrm{E}[C] = 20 + \frac{(1-p)^{30}}{2} \sum_{m=31}^{\infty} (m-30)(1-p)^{m-31} p. \tag{3}$$

Making the substitution $j = m - 30$ yields

$$\mathrm{E}[C] = 20 + \frac{(1-p)^{30}}{2} \sum_{j=1}^{\infty} j(1-p)^{j-1} p = 20 + \frac{(1-p)^{30}}{2p}. \tag{4}$$

Problem 4.2.4

In this problem, the CDF of W is

$$F_W(w) = \begin{cases} 0 & w < -5, \\ (w+5)/8 & -5 \leq w < -3, \\ 1/4 & -3 \leq w < 3, \\ 1/4 + 3(w-3)/8 & 3 \leq w < 5, \\ 1 & w \geq 5. \end{cases} \tag{1}$$

Each question can be answered directly from this CDF.

(a)

$$\mathrm{P}[W \leq 4] = F_W(4) = 1/4 + 3/8 = 5/8. \tag{2}$$

(b)

$$\mathrm{P}[-2 < W \leq 2] = F_W(2) - F_W(-2) = 1/4 - 1/4 = 0. \tag{3}$$

(c)

$$\mathrm{P}[W > 0] = 1 - \mathrm{P}[W \leq 0] = 1 - F_W(0) = 3/4. \tag{4}$$

(d) By inspection of $F_W(w)$, we observe that $\mathrm{P}[W \leq a] = F_W(a) = 1/2$ for a in the range $3 \leq a \leq 5$. In this range,

$$F_W(a) = 1/4 + 3(a-3)/8 = 1/2. \tag{5}$$

This implies $a = 11/3$.

Problem 4.3.6

$$f_X(x) = \begin{cases} ax^2 + bx & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

First, we note that a and b must be chosen such that the above PDF integrates to 1.

$$\int_0^1 (ax^2 + bx)\,dx = a/3 + b/2 = 1 \quad (2)$$

Hence, $b = 2 - 2a/3$ and our PDF becomes

$$f_X(x) = x(ax + 2 - 2a/3) \quad (3)$$

For the PDF to be non-negative for $x \in [0,1]$, we must have $ax + 2 - 2a/3 \ge 0$ for all $x \in [0,1]$. This requirement can be written as

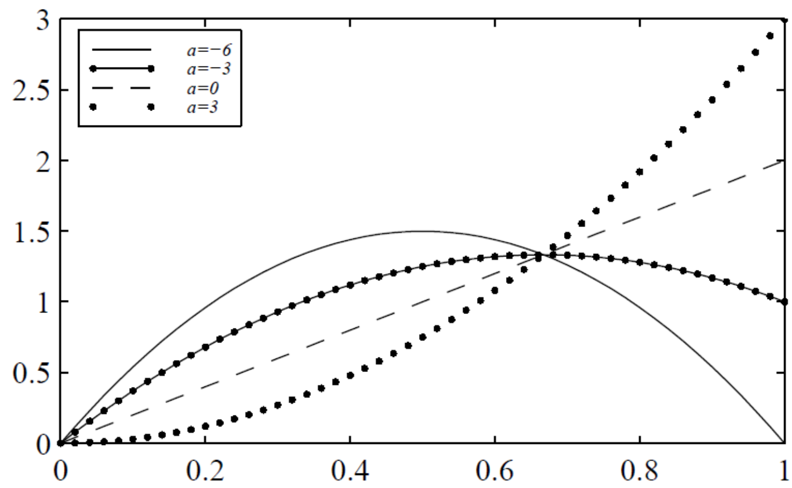
$$a(2/3 - x) \le 2, \qquad 0 \le x \le 1. \quad (4)$$

For $x = 2/3$, the requirement holds for all a. However, the problem is tricky because we must consider the cases $0 \le x < 2/3$ and $2/3 < x \le 1$ separately because of the sign change of the inequality. When $0 \le x < 2/3$, we have $2/3 - x > 0$ and the requirement is most stringent at $x = 0$ where we require $2a/3 \le 2$ or $a \le 3$. When $2/3 < x \le 1$, we can write the constraint as $a(x - 2/3) \ge -2$. In this case, the constraint is most stringent at $x = 1$, where we must have $a/3 \ge -2$ or $a \ge -6$. [Continued]

Thus a complete expression for our requirements are

$$-6 \le a \le 3, \qquad b = 2 - 2a/3. \quad (5)$$

As we see in the following plot, the shape of the PDF $f_X(x)$ varies greatly with the value of a.



Problem 4.4.7

To find the moments, we first find the PDF of U by taking the derivative of $F_U(u)$. The CDF and corresponding PDF are

$$F_U(u) = \begin{cases} 0 & u < -5, \\ (u+5)/8 & -5 \leq u < -3, \\ 1/4 & -3 \leq u < 3, \\ 1/4 + 3(u-3)/8 & 3 \leq u < 5, \\ 1 & u \geq 5. \end{cases} \tag{1}$$

$$f_U(u) = \begin{cases} 0 & u < -5, \\ 1/8 & -5 \leq u < -3, \\ 0 & -3 \leq u < 3, \\ 3/8 & 3 \leq u < 5, \\ 0 & u \geq 5. \end{cases} \tag{2}$$

(a) The expected value of U is

$$\begin{aligned} \mathrm{E}[U] = \int_{-\infty}^{\infty} u f_U(u)\, du &= \int_{-5}^{-3} \frac{u}{8}\, du + \int_{3}^{5} \frac{3u}{8}\, du \\ &= \left.\frac{u^2}{16}\right|_{-5}^{-3} + \left.\frac{3u^2}{16}\right|_{3}^{5} = 2. \end{aligned} \tag{3}$$

(b) The second moment of U is

$$\begin{aligned} \mathrm{E}\left[U^2\right] = \int_{-\infty}^{\infty} u^2 f_U(u)\, du &= \int_{-5}^{-3} \frac{u^2}{8}\, du + \int_{3}^{5} \frac{3u^2}{8}\, du \\ &= \left.\frac{u^3}{24}\right|_{-5}^{-3} + \left.\frac{u^3}{8}\right|_{3}^{5} = 49/3. \end{aligned} \tag{4}$$

The variance of U is $\mathrm{Var}[U] = \mathrm{E}[U^2] - (\mathrm{E}[U])^2 = 37/3$.

(c) Note that $2^U = e^{(\ln 2)U}$. This implies that

$$\int 2^u\, du = \int e^{(\ln 2)u}\, du = \frac{1}{\ln 2} e^{(\ln 2)u} = \frac{2^u}{\ln 2}. \tag{5}$$

The expected value of 2^U is then

$$\begin{aligned} \mathrm{E}\left[2^U\right] &= \int_{-\infty}^{\infty} 2^u f_U(u)\, du \\ &= \int_{-5}^{-3} \frac{2^u}{8}\, du + \int_{3}^{5} \frac{3 \cdot 2^u}{8}\, du \\ &= \left.\frac{2^u}{8 \ln 2}\right|_{-5}^{-3} + \left.\frac{3 \cdot 2^u}{8 \ln 2}\right|_{3}^{5} = \frac{2307}{256 \ln 2} = 13.001. \end{aligned} \tag{6}$$

Problem 4.5.8

Since U is continuous uniform and zero mean, we know that U is a $(-c, c)$ continuous uniform random variable with PDF

$$f_U(u) = \begin{cases} \frac{1}{2c} & -c \leq U \leq c, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

for some constant c. We also know that $\text{Var}[U] = (2c)^2/12 = c^2/3$. Thus,

$$\begin{aligned} \text{P}\left[U^2 \leq \text{Var}[U]\right] &= \text{P}\left[U^2 \leq c^2/3\right] \\ &= \text{P}\left[-c/\sqrt{3} \leq U \leq c/\sqrt{3}\right] \\ &= \int_{-c/\sqrt{3}}^{c/\sqrt{3}} \frac{1}{2c}\, du \\ &= \frac{1}{2c}\frac{2c}{\sqrt{3}} = \frac{1}{\sqrt{3}} = 0.5774. \end{aligned} \tag{2}$$

Problem 4.6.6

Using σ_T to denote the (unknown) standard deviation of T, we can write

$$\begin{aligned} \text{P}[T < 66] &= \text{P}\left[\frac{T-68}{\sigma_T} < \frac{66-68}{\sigma_T}\right] \\ &= \Phi\left(\frac{-2}{\sigma_T}\right) = 1 - \Phi\left(\frac{2}{\sigma_T}\right) = 0.1587. \end{aligned} \tag{1}$$

Thus $\Phi(2/\sigma_T) = 0.8413 = \Phi(1)$. This implies $\sigma_T = 2$ and thus T has variance $\text{Var}[T] = 4$.

Problem 4.7.5

The PMF of a geometric random variable with mean $1/p$ is

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The corresponding PDF is

$$\begin{aligned} f_X(x) &= p\delta(x-1) + p(1-p)\delta(x-2) + \cdots \\ &= \sum_{j=1}^{\infty} p(1-p)^{j-1}\delta(x-j). \end{aligned} \tag{2}$$

Additional Problem

[Poisson Arrival/Poisson process definition] ($N(t)$ is arrival count at time t)

I. $N(0) = 0$

II. The process has independent increments

$$P(N(t+s) - N(s) = n) = P(N(t) = n)$$

III. The number of events in any interval of length t is Poisson distributed with mean λt. That is for all $s, t \geq 0$

$$P(N(t+s) - N(s) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \quad n = 0,1,2$$

For inter-arrival time τ, at some arrival time s

$P(\tau \leq t) = 1 - P(\tau > t) = 1 - P(N(s+t) - N(s) = 0)$ (cause $\tau > t$ ⇔ no arrival at $(s, s+t)$)

$= 1 - P(N(t) - N(0) = 0)$ $by\ II$ independent increment by definition - *

$= 1 - P(N(t) = 0) = 1 - e^{-\lambda t}$ (is independent to s)

⇨ Inter-arrival time fallows exponential distribution