Problem 9.1.4

Each X_i has PDF identical to a random variable X. First we observe that $E[Y] = 3E[X] = 0$, implying $E[X] = 0$. Second, we observe that since the X_i are independent, Y has variance

$$\text{Var}[Y] = \text{Var}[X_1] + \text{Var}[X_2] + \text{Var}[X_3] = 3\,\text{Var}[X]. \tag{1}$$

Thus $\text{Var}[X] = \sigma_Y^2/3 = 4/3$. Finally, since X is a continuous uniform random variable, we need to find the parameters (a, b) which satisfy

$$E[X] = \frac{b+a}{2} = 0, \qquad \text{Var}[X] = \frac{(b-a)^2}{12} = \frac{4}{3}. \tag{2}$$

Thus $b + a = 0$ and $b - a = 4$, implying $b = 2$ and $a = -2$. The PDF of X_1 is thus

$$f_{X_1}(x) = f_X(x) = \begin{cases} 1/4 & -2 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Problem 9.2.4

Using the moment generating function of X, $\phi_X(s) = e^{\sigma^2 s^2/2}$. We can find the nth moment of X, $E[X^n]$ by taking the nth derivative of $\phi_X(s)$ and setting $s = 0$.

$$E[X] = \sigma^2 s e^{\sigma^2 s^2/2}\Big|_{s=0} = 0, \tag{1}$$

$$E\left[X^2\right] = \sigma^2 e^{\sigma^2 s^2/2} + \sigma^4 s^2 e^{\sigma^2 s^2/2}\Big|_{s=0} = \sigma^2. \tag{2}$$

Continuing in this manner we find that

$$E\left[X^3\right] = \left(3\sigma^4 s + \sigma^6 s^3\right) e^{\sigma^2 s^2/2}\Big|_{s=0} = 0, \tag{3}$$

$$E\left[X^4\right] = \left(3\sigma^4 + 6\sigma^6 s^2 + \sigma^8 s^4\right) e^{\sigma^2 s^2/2}\Big|_{s=0} = 3\sigma^4. \tag{4}$$

To calculate the moments of Y, we define $Y = X + \mu$ so that Y is Gaussian (μ, σ). In this case the second moment of Y is

$$E\left[Y^2\right] = E\left[(X+\mu)^2\right] = E\left[X^2 + 2\mu X + \mu^2\right] = \sigma^2 + \mu^2. \tag{5}$$

Similarly, the third moment of Y is

$$\begin{aligned} E\left[Y^3\right] &= E\left[(X+\mu)^3\right] \\ &= E\left[X^3 + 3\mu X^2 + 3\mu^2 X + \mu^3\right] = 3\mu\sigma^2 + \mu^3. \end{aligned} \tag{6}$$

Finally, the fourth moment of Y is

$$\begin{aligned} E\left[Y^4\right] &= E\left[(X+\mu)^4\right] \\ &= E\left[X^4 + 4\mu X^3 + 6\mu^2 X^2 + 4\mu^3 X + \mu^4\right] \\ &= 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4. \end{aligned} \tag{7}$$

Problem 9.3.3

In the iid random sequence $K_1, K_2, \ldots$, each K_i has PMF

$$P_K(k) = \begin{cases} 1-p & k=0, \\ p & k=1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) The MGF of K is $\phi_K(s) = \mathsf{E}[e^{sK}] = 1 - p + pe^s$.

(b) By Theorem 9.6, $M = K_1 + K_2 + \ldots + K_n$ has MGF

$$\phi_M(s) = [\phi_K(s)]^n = [1 - p + pe^s]^n . \tag{2}$$

(c) Although we could just use the fact that the expectation of the sum equals the sum of the expectations, the problem asks us to find the moments using $\phi_M(s)$. In this case,

$$\begin{aligned} \mathsf{E}[M] &= \left.\frac{d\phi_M(s)}{ds}\right|_{s=0} \\ &= n(1-p+pe^s)^{n-1}pe^s\big|_{s=0} = np. \end{aligned} \tag{3}$$

The second moment of M can be found via

$$\begin{aligned} \mathsf{E}\left[M^2\right] &= \left.\frac{d\phi_M(s)}{ds}\right|_{s=0} \\ &= np\left((n-1)(1-p+pe^s)pe^{2s} + (1-p+pe^s)^{n-1}e^s\right)\big|_{s=0} \\ &= np[(n-1)p+1]. \end{aligned} \tag{4}$$

The variance of M is

$$\mathsf{Var}[M] = \mathsf{E}\left[M^2\right] - (\mathsf{E}[M])^2 = np(1-p) = n\,\mathsf{Var}[K]. \tag{5}$$

Problem 9.4.3

(a) Let $X_1, \ldots, X_{120}$ denote the set of call durations (measured in minutes) during the month. From the problem statement, each $X - I$ is an exponential (λ) random variable with $\mathsf{E}[X_i] = 1/\lambda = 2.5$ min and $\mathsf{Var}[X_i] = 1/\lambda^2 = 6.25$ min^2. The total number of minutes used during the month is $Y = X_1 + \cdots + X_{120}$. By Theorem 9.1 and Theorem 9.3,

$$\begin{aligned} \mathsf{E}[Y] &= 120\,\mathsf{E}[X_i] = 300 \\ \mathsf{Var}[Y] &= 120\,\mathsf{Var}[X_i] = 750. \end{aligned} \tag{1}$$

The subscriber's bill is $30 + 0.4(y - 300)^+$ where $x^+ = x$ if $x \geq 0$ or $x^+ = 0$ if $x < 0$. the subscribers bill is exactly \$36 if $Y = 315$. The probability the subscribers bill exceeds \$36 equals

$$\begin{aligned} \mathsf{P}[Y > 315] &= \mathsf{P}\left[\frac{Y-300}{\sigma_Y} > \frac{315-300}{\sigma_Y}\right] \\ &= Q\left(\frac{15}{\sqrt{750}}\right) = 0.2919. \end{aligned} \tag{2}$$

(b) If the actual call duration is X_i, the subscriber is billed for $M_i = \lceil X_i \rceil$ minutes. Because each X_i is an exponential (λ) random variable, Theorem 4.9 says that M_i is a geometric (p) random variable with $p = 1 - e^{-\lambda} = 0.3297$. Since M_i is geometric,

$$E[M_i] = \frac{1}{p} = 3.033, \qquad \text{Var}[M_i] = \frac{1-p}{p^2} = 6.167. \tag{3}$$

The number of billed minutes in the month is $B = M_1 + \cdots + M_{120}$. Since $M_1, \ldots, M_{120}$ are iid random variables,

$$E[B] = 120\, E[M_i] = 364.0, \qquad \text{Var}[B] = 120\, \text{Var}[M_i] = 740.08. \tag{4}$$

Similar to part (a), the subscriber is billed \$36 if $B = 315$ minutes. The probability the subscriber is billed more than \$36 is

$$\begin{aligned} P[B > 315] &= P\left[\frac{B - 364}{\sqrt{740.08}} > \frac{315 - 365}{\sqrt{740.08}}\right] \\ &= Q(-1.8) = \Phi(1.8) = 0.964. \end{aligned} \tag{5}$$

Problem 10.1.3

This problem is in the wrong section since the *standard error* isn't defined until Section 10.4. However is we peek ahead to this section, the problem isn't very hard. Given the sample mean estimate $M_n(X)$, the standard error is defined as the standard deviation $e_n = \sqrt{\text{Var}[M_n(X)]}$. In our problem, we use samples X_i to generate $Y_i = X_i^2$. For the sample mean $M_n(Y)$, we need to find the standard error

$$e_n = \sqrt{\text{Var}[M_n(Y)]} = \sqrt{\frac{\text{Var}[Y]}{n}}. \tag{1}$$

Since X is a uniform $(0,1)$ random variable,

$$E[Y] = E[X^2] = \int_0^1 x^2\, dx = 1/3, \tag{2}$$

$$E[Y^2] = E[X^4] = \int_0^1 x^4\, dx = 1/5. \tag{3}$$

Thus $\text{Var}[Y] = 1/5 - (1/3)^2 = 4/45$ and the sample mean $M_n(Y)$ has standard error

$$e_n = \sqrt{\frac{4}{45n}}. \tag{4}$$

Problem 10.2.8

From Appendix A, we know that the MGF of K is

$$\phi_K(s) = e^{\alpha(e^s - 1)}. \tag{1}$$

The Chernoff bound becomes

$$P[K \geq c] \leq \min_{s \geq 0} e^{-sc} e^{\alpha(e^s-1)} = \min_{s \geq 0} e^{\alpha(e^s-1)-sc}. \tag{2}$$

Since e^y is an increasing function, it is sufficient to choose s to minimize $h(s) = \alpha(e^s - 1) - sc$. Setting $dh(s)/ds = \alpha e^s - c = 0$ yields $e^s = c/\alpha$ or $s = \ln(c/\alpha)$. Note that for $c < \alpha$, the minimizing s is negative. In this case, we choose $s = 0$ and the Chernoff bound is $P[K \geq c] \leq 1$. For $c \geq \alpha$, applying $s = \ln(c/\alpha)$ yields $P[K \geq c] \leq e^{-\alpha}(\alpha e/c)^c$. A complete expression for the Chernoff bound is

$$P[K \geq c] \leq \begin{cases} 1 & c < \alpha, \\ \alpha^c e^c e^{-\alpha}/c^c & c \geq \alpha. \end{cases} \tag{3}$$

Problem 10.3.4

Since the relative frequency of the error event E is $\hat{P}_n(E) = M_n(X_E)$ and $\mathrm{E}[M_n(X_E)] = \mathrm{P}[E]$, we can use Theorem 10.5(a) to write

$$\mathrm{P}\left[\left|\hat{P}_n(A) - \mathrm{P}\left[E\right]\right| \geq c\right] \leq \frac{\mathrm{Var}[X_E]}{nc^2}. \tag{1}$$

Note that $\mathrm{Var}[X_E] = \mathrm{P}[E](1 - \mathrm{P}[E])$ since X_E is a Bernoulli ($p = \mathrm{P}[E]$) random variable. Using the additional fact that $\mathrm{P}[E] \leq \epsilon$ and the fairly trivial fact that $1 - \mathrm{P}[E] \leq 1$, we can conclude that

$$\mathrm{Var}[X_E] = \mathrm{P}\left[E\right]\left(1 - \mathrm{P}\left[E\right]\right) \leq \mathrm{P}\left[E\right] \leq \epsilon. \tag{2}$$

Thus

$$\mathrm{P}\left[\left|\hat{P}_n(A) - \mathrm{P}\left[E\right]\right| \geq c\right] \leq \frac{\mathrm{Var}[X_E]}{nc^2} \leq \frac{\epsilon}{nc^2}. \tag{3}$$

Problem 10.4.4

(a) Since the expectation of a sum equals the sum of the expectations also holds for vectors,

$$\mathrm{E}\left[\mathbf{M}(n)\right] = \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}\left[\mathbf{X}(i)\right] = \frac{1}{n}\sum_{i=1}^{n} \mu_{\mathbf{X}} = \mu_{\mathbf{X}}. \tag{1}$$

(b) The jth component of $\mathbf{M}(n)$ is $M_j(n) = \frac{1}{n}\sum_{i=1}^{n} X_j(i)$, which is just the sample mean of X_j. Defining $A_j = \{|M_j(n) - \mu_j| \geq c\}$, we observe that

$$\mathrm{P}\left[\max_{j=1,\ldots,k} |M_j(n) - \mu_j| \geq c\right] = \mathrm{P}\left[A_1 \cup A_2 \cup \cdots \cup A_k\right]. \tag{2}$$

Applying the Chebyshev inequality to $M_j(n)$, we find that

$$\mathrm{P}\left[A_j\right] \leq \frac{\mathrm{Var}[M_j(n)]}{c^2} = \frac{\sigma_j^2}{nc^2}. \tag{3}$$

By the union bound,

$$\mathrm{P}\left[\max_{j=1,\ldots,k} |M_j(n) - \mu_j| \geq c\right] \leq \sum_{j=1}^{k} \mathrm{P}\left[A_j\right] \leq \frac{1}{nc^2}\sum_{j=1}^{k} \sigma_j^2. \tag{4}$$

Since $\sum_{j=1}^{k} \sigma_j^2 < \infty$, $\lim_{n\to\infty} \mathrm{P}[\max_{j=1,\ldots,k} |M_j(n) - \mu_j| \geq c] = 0$.

Problem 10.4.6

(a) Since the expectation of the sum equals the sum of the expectations,

$$\mathrm{E}\left[\hat{\mathbf{R}}(n)\right] = \frac{1}{n}\sum_{m=1}^{n} \mathrm{E}\left[\mathbf{X}(m)\mathbf{X}'(m)\right] = \frac{1}{n}\sum_{m=1}^{n} \mathbf{R} = \mathbf{R}. \quad (1)$$

(b) This proof follows the method used to solve Problem 10.4.4. The i,jth element of $\hat{\mathbf{R}}(n)$ is $\hat{R}_{i,j}(n) = \frac{1}{n}\sum_{m=1}^{n} X_i(m)X_j(m)$, which is just the sample mean of X_iX_j. Defining the event

$$A_{i,j} = \left\{\left|\hat{R}_{i,j}(n) - \mathrm{E}\left[X_iX_j\right]\right| \geq c\right\}, \quad (2)$$

we observe that

$$\mathrm{P}\left[\max_{i,j}\left|\hat{R}_{i,j}(n) - \mathrm{E}\left[X_iX_j\right]\right| \geq c\right] = \mathrm{P}\left[\cup_{i,j} A_{i,j}\right]. \quad (3)$$

Applying the Chebyshev inequality to $\hat{R}_{i,j}(n)$, we find that

$$\mathrm{P}\left[A_{i,j}\right] \leq \frac{\mathrm{Var}[\hat{R}_{i,j}(n)]}{c^2} = \frac{\mathrm{Var}[X_iX_j]}{nc^2}. \quad (4)$$

By the union bound,

$$\begin{aligned}\mathrm{P}\left[\max_{i,j}\left|\hat{R}_{i,j}(n) - \mathrm{E}\left[X_iX_j\right]\right| \geq c\right] &\leq \sum_{i,j} \mathrm{P}\left[A_{i,j}\right] \\ &\leq \frac{1}{nc^2}\sum_{i,j} \mathrm{Var}[X_iX_j]. \quad (5)\end{aligned}$$

By the result of Problem 7.6.4, X_iX_j, the product of jointly Gaussian random variables, has finite variance. Thus

$$\begin{aligned}\sum_{i,j} \mathrm{Var}[X_iX_j] &= \sum_{i=1}^{k}\sum_{j=1}^{k} \mathrm{Var}[X_iX_j] \\ &\leq k^2 \max_{i,j} \mathrm{Var}[X_iX_j] < \infty. \quad (6)\end{aligned}$$

It follows that

$$\begin{aligned}\lim_{n\to\infty} \mathrm{P}\left[\max_{i,j}\left|\hat{R}_{i,j}(n) - \mathrm{E}\left[X_iX - j\right]\right| \geq c\right] &\leq \lim_{n\to\infty} \frac{k^2 \max_{i,j} \mathrm{Var}[X_iX_j]}{nc^2} \\ &= 0. \quad (7)\end{aligned}$$