Problem 1.1.2

(a) An outcome specifies whether the connection speed is high (h), medium (m), or low (l) speed, and whether the signal is a mouse click (c) or a tweet (t). The sample space is

$$S = \{ht, hc, mt, mc, lt, lc\}. \tag{1}$$

(b) The event that the wi-fi connection is medium speed is $A_1 = \{mt, mc\}$.

(c) The event that a signal is a mouse click is $A_2 = \{hc, mc, lc\}$.

(d) The event that a connection is either high speed or low speed is $A_3 = \{ht, hc, lt, lc\}$.

(e) Since $A_1 \cap A_2 = \{mc\}$ and is not empty, A_1, A_2, and A_3 are not mutually exclusive.

(f) Since

$$A_1 \cup A_2 \cup A_3 = \{ht, hc, mt, mc, lt, lc\} = S, \tag{2}$$

the collection A_1, A_2, A_3 is collectively exhaustive.

Problem 1.2.12

It is tempting to use the following proof:

Since S and ϕ are mutually exclusive, and since $S = S \cup \phi$,

$$1 = \mathrm{P}[S \cup \phi] = \mathrm{P}[S] + \mathrm{P}[\phi].$$

Since $\mathrm{P}[S] = 1$, we must have $\mathrm{P}[\phi] = 0$.

The above "proof" used the property that for mutually exclusive sets A_1 and A_2,

$$\mathrm{P}[A_1 \cup A_2] = \mathrm{P}[A_1] + \mathrm{P}[A_2]. \tag{1}$$

The problem is that this property is a consequence of the three axioms, and thus must be proven. [Continued]

For a proof that uses just the three axioms, let A_1 be an arbitrary set and for $n = 2, 3, \ldots$, let $A_n = \phi$. Since $A_1 = \cup_{i=1}^{\infty} A_i$, we can use Axiom 3 to write

$$\mathrm{P}[A_1] = \mathrm{P}[\cup_{i=1}^{\infty} A_i] = \mathrm{P}[A_1] + \mathrm{P}[A_2] + \sum_{i=3}^{\infty} \mathrm{P}[A_i]. \qquad (2)$$

By subtracting P$[A_1]$ from both sides, the fact that $A_2 = \phi$ permits us to write

$$\mathrm{P}[\phi] + \sum_{n=3}^{\infty} \mathrm{P}[A_i] = 0. \qquad (3)$$

By Axiom 1, P$[A_i] \geq 0$ for all i. Thus, $\sum_{n=3}^{\infty} \mathrm{P}[A_i] \geq 0$. This implies P$[\phi] \leq 0$. Since Axiom 1 requires P$[\phi] \geq 0$, we must have P$[\phi] = 0$.

Problem 1.3.2

Let s_i denote the outcome that the roll is i. So, for $1 \leq i \leq 6$, $R_i = \{s_i\}$. Similarly, $G_j = \{s_{j+1}, \ldots, s_6\}$.

(a) Since $G_1 = \{s_2, s_3, s_4, s_5, s_6\}$ and all outcomes have probability 1/6, P$[G_1] = 5/6$. The event $R_3G_1 = \{s_3\}$ and P$[R_3G_1] = 1/6$ so that

$$\mathrm{P}[R_3|G_1] = \frac{\mathrm{P}[R_3G_1]}{\mathrm{P}[G_1]} = \frac{1}{5}. \qquad (1)$$

(b) The conditional probability that 6 is rolled given that the roll is greater than 3 is

$$\mathrm{P}[R_6|G_3] = \frac{\mathrm{P}[R_6G_3]}{\mathrm{P}[G_3]} = \frac{\mathrm{P}[s_6]}{\mathrm{P}[s_4, s_5, s_6]} = \frac{1/6}{3/6}. \qquad (2)$$

(c) The event E that the roll is even is $E = \{s_2, s_4, s_6\}$ and has probability 3/6. The joint probability of G_3 and E is

$$\mathrm{P}[G_3E] = \mathrm{P}[s_4, s_6] = 1/3. \qquad (3)$$

The conditional probabilities of G_3 given E is

$$\mathrm{P}[G_3|E] = \frac{\mathrm{P}[G_3E]}{\mathrm{P}[E]} = \frac{1/3}{1/2} = \frac{2}{3}. \qquad (4)$$

(d) The conditional probability that the roll is even given that it's greater than 3 is

$$\mathrm{P}[E|G_3] = \frac{\mathrm{P}[EG_3]}{\mathrm{P}[G_3]} = \frac{1/3}{1/2} = \frac{2}{3}. \qquad (5)$$

Problem 1.4.2

(a) From the given probability distribution of billed minutes, M, the probability that a call is billed for more than 3 minutes is

$$\begin{aligned} P[L] &= 1 - P[3 \text{ or fewer billed minutes}] \\ &= 1 - P[B_1] - P[B_2] - P[B_3] \\ &= 1 - \alpha - \alpha(1-\alpha) - \alpha(1-\alpha)^2 \\ &= (1-\alpha)^3 = 0.57. \end{aligned} \tag{1}$$

(b) The probability that a call will billed for 9 minutes or less is

$$P[9 \text{ minutes or less}] = \sum_{i=1}^{9} \alpha(1-\alpha)^{i-1} = 1 - (0.57)^3. \tag{2}$$

Problem 1.5.3

Let A_i and B_i denote the events that the ith phone sold is an Apricot or a Banana respectively. The works "each phone sold is twice as likely to be an Apricot than a Banana" tells us that

$$P[A_i] = 2\,P[B_i]. \tag{1}$$

However, since each phone sold is either an Apricot or a Banana, A_i and B_i are a partition and

$$P[A_i] + P[B_i] = 1. \tag{2}$$

Combining these equations, we have $P[A_i] = 2/3$ and $P[B_i] = 1/3$. The probability that two phones sold are the same is

$$P[A_1A_2 \cup B_1B_2] = P[A_1A_2] + P[B_1B_2]. \tag{3}$$

Since "each phone sale is independent,"

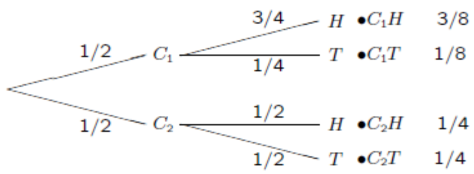
$$P[A_1A_2] = P[A_1]\,P[A_2] = \frac{4}{9}, \qquad P[B_1B_2] = P[B_1]\,P[B_2] = \frac{1}{9}. \tag{4}$$

Thus the probability that two phones sold are the same is

$$P[A_1A_2 \cup B_1B_2] = P[A_1A_2] + P[B_1B_2] = \frac{4}{9} + \frac{1}{9} = \frac{5}{9}. \tag{5}$$

Problem 2.1.5

Here is the tree diagram for the experiment:



To find the conditional probabilities, we see

$$\begin{aligned}\mathrm{P}[C_1|H] &= \frac{\mathrm{P}[C_1H]}{\mathrm{P}[H]} \\ &= \frac{\mathrm{P}[C_1H]}{\mathrm{P}[C_1H]+\mathrm{P}[C_2H]}\end{aligned}$$

From the leaf probabilities in the sample tree,

$$\mathrm{P}[C_1|H] = \frac{3/8}{3/8+1/4} = \frac{3}{5}.$$

Similarly,

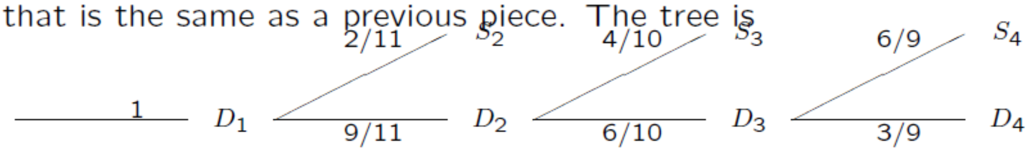
$$\mathrm{P}[C_1|T] = \frac{\mathrm{P}[C_1T]}{\mathrm{P}[T]} = \frac{\mathrm{P}[C_1T]}{\mathrm{P}[C_1T]+\mathrm{P}[C_2T]} = \frac{1/8}{1/8+1/4} = \frac{1}{3}. \qquad (1)$$

As we would expect, we are more likely to have chosen coin 1 when the first flip is heads, but we are more likely to have chosen coin 2 when the first flip is tails.

Problem 2.2.3

(a) Since there are only three pieces of each flavor, we cannot draw four pieces of all the same flavor. Hence $P[F_1] = 0$.

(b) Let D_i denote the event that the ith piece is a different flavor from all the prior pieces. Let S_i denote the event that piece i is the same flavor as a previous piece. A tree for this experiment is relatively simple because we stop the experiment as soon as we draw a piece that is the same as a previous piece. The tree is



Note that:

- $P[D_1] = 1$ because the first piece is "different" since there haven't been any prior pieces.
- For the second piece, there are 11 pieces left and 9 of those pieces are different from the first piece drawn.
- Given the first two pieces are different, there are 2 colors, each with 3 pieces (6 pieces) out of 10 remaining pieces that are a different flavor from the first two pieces. Thus $P[D_3|D_2D_1] = 6/10$.
- Finally, given the first three pieces are different flavors, there are 3 pieces remaining that are a different flavor from the pieces previously picked.

Thus $P[D_4|D_3D_2D_1] = 3/9$. It follows that the three pieces are different with probability

$$P[D_1D_2D_3D_4] = 1\left(\frac{9}{11}\right)\left(\frac{6}{10}\right)\frac{3}{9} = \frac{9}{55}. \tag{1}$$

An alternate approach to this problem is to assume that each piece is distinguishable, say by numbering the pieces $1, 2, 3$ in each flavor. In addition, we define the outcome of the experiment to be a 4-permutation of the 12 distinguishable pieces. Under this model, there are $(12)_4 = \frac{12!}{8!}$ equally likely outcomes in the sample space. The number of outcomes in which all four pieces are different is $n_4 = 12 \cdot 9 \cdot 6 \cdot 3$ since there are 12 choices for the first piece drawn, 9 choices for the second piece from the three remaining flavors, 6 choices for the third piece and three choices for the last piece. Since all outcomes are equally likely,

$$P[F_4] = \frac{n_4}{(12)_4} = \frac{12 \cdot 9 \cdot 6 \cdot 3}{12 \cdot 11 \cdot 10 \cdot 9} = \frac{9}{55} \tag{2}$$

(c) The second model of distinguishable starburst pieces makes it easier to solve this last question. In this case, let the outcome of the experiment be the $\binom{12}{4} = 495$ combinations or pieces. In this case, we are ignoring the order in which the pieces were selected. Now we count the number of combinations in which we have two pieces of each of two flavors. We can do this with the following steps:

1. Choose two of the four flavors.
2. Choose 2 out of 3 pieces of one of the two chosen flavors.
3. Choose 2 out of 3 pieces of the other of the two chosen flavors.

Let n_i equal the number of ways to execute step i. We see that

$$n_1 = \binom{4}{2} = 6, \qquad n_2 = \binom{3}{2} = 3, \qquad n_3 = \binom{3}{2} = 3. \tag{3}$$

The number of possible ways to execute this sequence of steps is $n_1 n_2 n_3 = 54$ Since all combinations are equally likely,

$$\mathrm{P}[F_2] = \frac{n_1 n_2 n_3}{\binom{12}{4}} = \frac{54}{495} = \frac{6}{55}. \tag{4}$$

Problem 2.2.10

(a) This is just the multinomial probability

$$\begin{aligned}\mathrm{P}[A] &= \binom{40}{19, 19, 2} \left(\frac{19}{40}\right)^{19} \left(\frac{19}{40}\right)^{19} \left(\frac{2}{40}\right)^{2} \\ &= \frac{40!}{19!19!2!} \left(\frac{19}{40}\right)^{19} \left(\frac{19}{40}\right)^{19} \left(\frac{2}{40}\right)^{2}. \end{aligned} \tag{1}$$

(b) Each spin is either green (with probability 19/40) or not (with probability 21/40). If we call landing on greeen a success, then G_{19} is the probability of 19 successes in 40 trials. Thus

$$\mathrm{P}[G_{19}] = \binom{40}{19} \left(\frac{19}{40}\right)^{19} \left(\frac{21}{40}\right)^{21}. \tag{2}$$

(c) If you bet on red, the probability you win is 19/40. If you bet green, the probability that you win is 19/40. If you first make a random choice to bet red or green, (say by flipping a coin), the probability you win is still $p = 19/40$.

Problem 2.3.5

(a) There are 3 group 1 kickers and 6 group 2 kickers. Using G_i to denote that a group i kicker was chosen, we have

$$P[G_1] = 1/3, \qquad P[G_2] = 2/3. \tag{1}$$

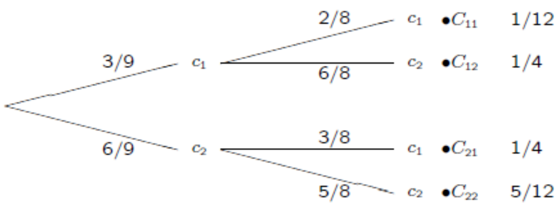
In addition, the problem statement tells us that

$$P[K|G_1] = 1/2, \qquad P[K|G_2] = 1/3. \tag{2}$$

Combining these facts using the Law of Total Probability yields

$$\begin{aligned} P[K] &= P[K|G_1]\,P[G_1] + P[K|G_2]\,P[G_2] \\ &= (1/2)(1/3) + (1/3)(2/3) = 7/18. \end{aligned} \tag{3}$$

(b) To solve this part, we need to identify the groups from which the first and second kicker were chosen. Let c_i indicate whether a kicker was chosen from group i and let C_{ij} indicate that the first kicker was chosen from group i and the second kicker from group j. The experiment to choose the kickers is described by the sample tree:



Since a kicker from group 1 makes a kick with probability 1/2 while a kicker from group 2 makes a kick with probability 1/3,

$$P[K_1K_2|C_{11}] = (1/2)^2, \qquad P[K_1K_2|C_{12}] = (1/2)(1/3), \tag{4}$$

$$P[K_1K_2|C_{21}] = (1/3)(1/2), \qquad P[K_1K_2|C_{22}] = (1/3)^2. \tag{5}$$

By the law of total probability,

$$\begin{aligned} P[K_1K_2] &= P[K_1K_2|C_{11}]\,P[C_{11}] + P[K_1K_2|C_{12}]\,P[C_{12}] \\ &\quad + P[K_1K_2|C_{21}]\,P[C_{21}] + P[K_1K_2|C_{22}]\,P[C_{22}] \\ &= \frac{1}{4}\frac{1}{12} + \frac{1}{6}\frac{1}{4} + \frac{1}{6}\frac{1}{4} + \frac{1}{9}\frac{5}{12} = 15/96. \end{aligned} \tag{6}$$

It should be apparent that $P[K_1] = P[K]$ from part (a). Symmetry should also make it clear that $P[K_1] = P[K_2]$ since for any ordering of two kickers, the reverse ordering is equally likely. If this is not clear, we derive this result by calculating $P[K_2|C_{ij}]$ and using the law of total probability to calculate $P[K_2]$.

$$P[K_2|C_{11}] = 1/2, \qquad P[K_2|C_{12}] = 1/3, \tag{7}$$

$$P[K_2|C_{21}] = 1/2, \qquad P[K_2|C_{22}] = 1/3. \tag{8}$$

By the law of total probability,

$$\begin{aligned} P[K_2] &= P[K_2|C_{11}]\,P[C_{11}] + P[K_2|C_{12}]\,P[C_{12}] \\ &\quad + P[K_2|C_{21}]\,P[C_{21}] + P[K_2|C_{22}]\,P[C_{22}] \\ &= \frac{1}{2}\frac{1}{12} + \frac{1}{3}\frac{1}{4} + \frac{1}{2}\frac{1}{4} + \frac{1}{3}\frac{5}{12} = \frac{7}{18}. \end{aligned} \tag{9}$$

We observe that K_1 and K_2 are not independent since

$$P[K_1K_2] = \frac{15}{96} \neq \left(\frac{7}{18}\right)^2 = P[K_1]\,P[K_2]. \tag{10}$$

Note that 15/96 and $(7/18)^2$ are close but not exactly the same. The reason K_1 and K_2 are dependent is that if the first kicker is successful, then it is more likely that kicker is from group 1. This makes it more likely that the second kicker is from group 2 and is thus more likely to miss.

(c) Once a kicker is chosen, each of the 10 field goals is an independent trial. If the kicker is from group 1, then the success probability is 1/2. If the kicker is from group 2, the success probability is 1/3. Out of 10 kicks, there are 5 misses iff there are 5 successful kicks. Given the type of kicker chosen, the probability of 5 misses is

$$\mathrm{P}\left[M|G_1\right] = \binom{10}{5}(1/2)^5(1/2)^5, \qquad \mathrm{P}\left[M|G_2\right] = \binom{10}{5}(1/3)^5(2/3)^5. \tag{11}$$

We use the Law of Total Probability to find

$$\begin{aligned} \mathrm{P}\left[M\right] &= \mathrm{P}\left[M|G_1\right]\mathrm{P}\left[G_1\right] + \mathrm{P}\left[M|G_2\right]\mathrm{P}\left[G_2\right] \\ &= \binom{10}{5}\left((1/3)(1/2)^{10} + (2/3)(1/3)^5(2/3)^5\right). \end{aligned} \tag{12}$$